International Journal of Multiphysics
Volume 18, No. 3, 2024
ISSN: 1750-9548

Alpha Logarithm Lomax Distribution with Application

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Abstract

In this paper, we present a new distribution called the Alpha Logarithm Lomax (ALL) distribution by using the Lomax distribution with the alpha Log-G distributions family; the new distribution has three parameters and provides greater flexibility than the Lomax distribution, which is well-known for modeling survival and lifespan data. We investigate several properties of the ALL distribution, including its moment-generating function and various statistical properties, and explore the maximum likelihood estimation (MLE) method to estimate the three parameters of this distribution. We use simulations to demonstrate the effect of MLE, and the new distribution is applied to accurate data to prove its superiority over some other distributions for these data.

Keywords: Lomax distribution, moment generating function, maximum likelihood estimation, alpha Log-G distributions family.

1. Introduction

The Lomax distribution constitutes a heavy-tailed continuous distribution delineated by a singular shape parameter alongside a scale parameter. Lomax initially introduced it in 1954 to analyze data about business failures [1]. The Lomax distribution relevance encompasses modeling heavy-tailed data, as in wealth and income distributions, business environments, and other sciences. Investigators have also delved into various methodologies aimed at constructing generalized families of distributions and extending the Lomax distribution by incorporating one, two, or even three additional parameters, thereby enhancing its adaptability to more accurately represent diverse categories of accurate data. For instance, In the latest studies, Bakoban et al. introduced the Alpha Power Lomax distribution as a new distribution that encompasses characteristics such as mode, quantiles, and entropies, with parameters estimated utilizing multiple methods—a simulation study of the efficacy of estimators [2]. Kamal et al. showcased the Arc-Sine Exponentiation Lomax Distribution, a new model that captures diverse density and hazard function characteristics, making it applicable to economic studies [3]. Telee et al. proposed the Modified Lomax Distribution, which has three parameters alongside statistical properties and estimation methodologies such as Least Squares, Cramer-Von Mises, and Maximum Likelihood [4]. Abdullah et al. revealed the New Alpha Power Transformed Power Lomax Distribution. This four-parameter distribution is developed through an alpha power transformation approach. It is highly capable of interpreting lifetime data collections, employing parameter estimation through maximum likelihood and the Bayesian method [5].

L. Mohsin and H. Kalt introduced the Alpha Log-G (ALG) family of distributions [6], representing a novel methodological framework that integrates an additional parameter into the distribution function. This family is fundamentally anchored in the logarithmic function linked to the newly added parameter derived from the essential continuous distribution. The distribution function (CDF) for this Alpha log-G family was illustrated as follows

$$F_{ALG}(x;\alpha,\varphi)=\frac{\log(\alpha G(x;\varphi)+1)}{\log(\alpha+1)},0<\alpha \tag{1}$$

and the probability density function of the Alpha Log-G family is

$$f_{ALG}(x;\alpha,\varphi)=\frac{\alpha g(x;\varphi)}{\log(\alpha+1)(\alpha G(x;\varphi)+1)} \tag{2}$$

Where $G(x;\varphi)$ and $g(x;\varphi)$ are the CDF and PDF of the baseline distributing, respectively, with φ being the victor of parameters.

This paper seeks to establish a novel continuous probability distribution designated as the Alpha Logarithm Lomax (ALL) distribution, delineated by three parameters, utilizing the ALG family as its foundational framework. This objective is accomplished by incorporating an additional parameter that significantly augments its adaptability within the (ALG) family structure. The organization of this study is methodically arranged as follows: The second section is dedicated to identifying the newly proposed distribution alongside the formulation and empirical validation of its (CDF) and (PDF), utilizing graphical illustrations across a spectrum of parameter values. In the third section, the emphasis shifts to the exploration and computation of vital statistical functions associated with the newly developed distribution, including instances, maximum likelihood estimators, and structured data analyses. The fourth section is devoted to an in-depth analysis of estimator simulations, aiming to evaluate their convergence towards actual values as variations in sample sizes are implemented. Finally, the fifth section addresses the application of the novel distribution to three distinct empirical datasets, juxtaposed with previously examined distributions applied to the same datasets, to substantiate the suitability of the innovative distribution for these data sets and to establish its superiority compared to alternative distributions. The technique of generalizing distributions is of great interest to researchers in the field of statistics, and the essential functions of new distributions and their related properties are often derived [7,8,9,10], as this research deals with.

2. The Alpha Logarithm Lomax (ALL) distribution

Given the significant relevance of the Lomax distribution, it has been utilized in conjunction with the logarithmic transformation through the ALG family. The new distribution is called the Alpha Logarithm Lomax (ALL) distribution. By substituting the CDF of Lomax distribution into formula (1), we get:

$$F(x;\alpha,\lambda,\theta)=\frac{ln\left(\alpha-\alpha\left[1+\frac{x}{\lambda}\right]^{-\theta}+1\right)}{ln(\alpha+1)} \qquad (3)$$

Equation (3) represents the cumulative distribution function for the ALL distribution. Now, we derive equation (3) to extract the density function for the ALL distribution.

$$f(x;\alpha,\lambda,\theta)=\frac{\alpha\theta\left[1+\frac{x}{\lambda}\right]^{-(\theta+1)}}{\lambda ln(\alpha+1)\left[\alpha-\alpha\left[1+\frac{x}{\lambda}\right]^{-\theta}+1\right]} \quad ,0<x<\infty \ ,\alpha,\lambda,\theta>0, \qquad (4)$$

Where α and λ are the shape parameters, the following figures show the plots of different parameter values chosen for (PDF) and (CDF) of the ALL distribution for various parameter values.

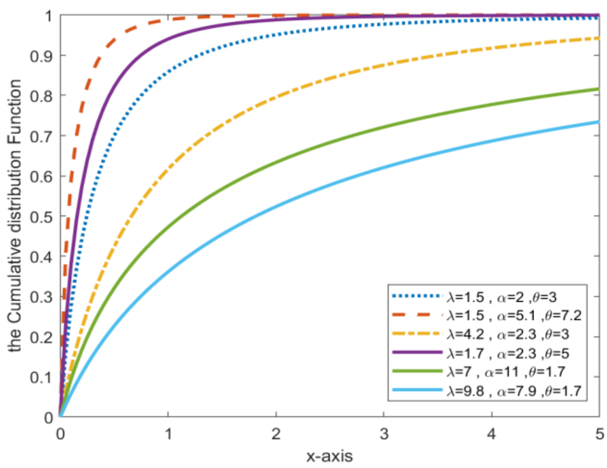

Figure (1): the CDF plot of ALL distribution.

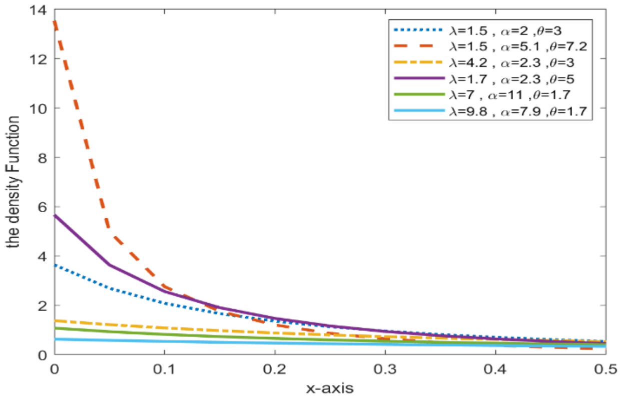

Figure (2): The PDF plot of ALL distribution.

3. The Statistical Functions

we introduce the survival (reliability) function $\bar{F}(x\colon \alpha,\lambda,\theta)$, the hazard rate function $h(x\colon \alpha,\lambda,\theta)$, the reversed hazard function $r(x\colon \alpha,\lambda,\theta)$ and the cumulative function of the hazard rate function $H(x\colon \alpha,\lambda,\theta)$ Of ALL distributing [11,12].

3.1 Survival function of ALL distribution

The survival function $\bar{F}(x\colon \alpha,\lambda,\theta)$ of $X\sim ALL(\alpha,\lambda,\theta)$ is defined as follows

$$\bar{F}(x\colon \alpha,\lambda,\theta) = 1 - F(x\colon \alpha,\lambda,\theta) = \frac{ln(\alpha+1) - ln(\alpha - \alpha\left[1+\frac{x}{\lambda}\right]^{-\theta} + 1)}{ln(\alpha+1)}. \qquad (5)$$

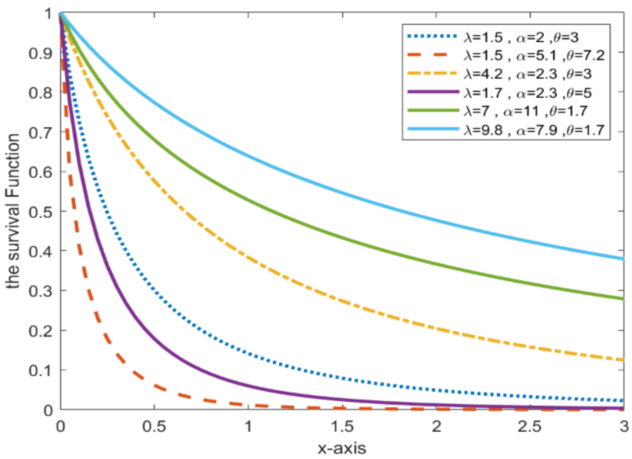

Figure (3): The survival function of ALL distribution.

3.2 Hazard function of ALL distribution

The hazard function $h(x\colon \alpha,\lambda,\theta)$ of the ALL distributing is defined as follows

$$h(x\colon \alpha,\lambda,\theta) = \frac{f(x\colon \alpha,\lambda,\theta)}{\bar{F}(x\colon \alpha,\lambda,\theta)} = \frac{\alpha\theta\left[1+\frac{x}{\lambda}\right]^{-(\theta+1)}}{\lambda(\alpha - \alpha\left[1+\frac{x}{\lambda}\right]^{-\theta} + 1)ln\left[\frac{\alpha+1}{\alpha - \alpha\left[1+\frac{x}{\lambda}\right]^{-\theta} + 1}\right]} \qquad (6)$$

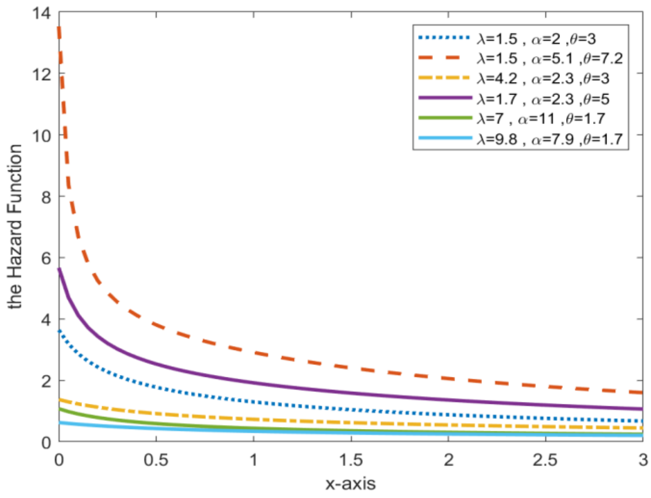

Figure (4): The Hazard function of ALL distribution.

3.3 The reverse hazard function of ALL distribution

The reverse hazard function $\boldsymbol{r(x:\alpha,\lambda,\theta)}$ of ALL distributing is defined as follows

$$\boldsymbol{r(x:\alpha,\lambda,\theta)}=\frac{\boldsymbol{f(x:\alpha,\lambda,\theta)}}{\boldsymbol{F(x:\alpha,\lambda,\theta)}}=\frac{\alpha\theta\left[1+\frac{x}{\lambda}\right]^{-(\theta+1)}}{\lambda(\alpha-\alpha\left[1+\frac{x}{\lambda}\right]^{-\theta}+1)ln\left[\alpha-\alpha\left[1+\frac{x}{\lambda}\right]^{-\theta}+1\right]} \tag{7}$$

3.4 The cumulative hazard function

The cumulative hazard $H(x:\alpha,\lambda,\theta)$ of ALL distribution is defined as

$$H(x:\alpha,\lambda,\theta)=-Ln[1-G(x:\alpha,\lambda,\theta)]=-Ln\left[\frac{ln(\alpha+1)-ln(\alpha-\alpha\left[1+\frac{x}{\lambda}\right]^{-\theta}+1)}{ln(\alpha+1)}\right] \tag{8}$$

3.5 The Moments

Here, we will discuss the $\boldsymbol{r^{th}}$ moment about the origin of ALL distributing[13].

Theorem 1. Let X be the random variable of the ALL distributing, then the r^{th} moment function (moments at the origin point) is given by:

$$\grave{\mu}_r=\frac{\lambda^r\theta}{\ln(\alpha+1)}\sum_{n=0}^{\infty}\binom{r}{n}(-1)^{(r-n)}\sum_{k=0}^{\infty}\frac{\left(\frac{\alpha}{\alpha+1}\right)^{k+1}}{\theta(1+k)-n} \tag{9}$$

Proof: We will begin with the definition of the r^{th}moment function at the origin point of the random variable X with (PDF) of the Lomax Distributing ALL distributing given by:

$$\grave{\mu}_r=E(x^r)=\int_0^{\infty}x^r\ f(x:\alpha,\lambda,\theta)\ dx,\qquad r=1,2,3,\ldots$$

$$=\int_0^{\infty}x^r\,\frac{\alpha\theta\left[1+\frac{x}{\lambda}\right]^{-(\theta+1)}}{\lambda ln(\alpha+1)\left[\alpha-\alpha\left[1+\frac{x}{\lambda}\right]^{-\theta}+1\right]}dx,$$

$$= \frac{\alpha\theta}{\lambda ln(\alpha+1)} \int_0^\infty x^r \frac{\left[1+\frac{x}{\lambda}\right]^{-(\theta+1)}}{\left[\alpha - \alpha\left[1+\frac{x}{\lambda}\right]^{-\theta} + 1\right]} dx,$$

$$= \frac{\alpha\theta}{\lambda ln(\alpha+1)} \int_0^\infty \frac{x^r}{(\alpha+1)\left[1+\frac{x}{\lambda}\right]^{\theta+1} - \alpha\left[1+\frac{x}{\lambda}\right]} dx, \tag{10}$$

Let y=1 $+\frac{x}{\lambda} \rightarrow x = \lambda(y-1) \rightarrow dx = \lambda dy$, then

$$\grave{\mu}_r = \frac{\lambda^r \alpha\theta}{ln(\alpha+1)} \int_1^\infty \frac{(y-1)^r}{(\alpha+1)y^{\theta+1} - \alpha y} dy,$$

Since $(y-1)^r = \sum_{n=0}^{\infty} \binom{r}{n} (-1)^{(r-n)} y^n$, so we have

$$\grave{\mu}_r = \frac{\lambda^r \alpha\theta}{ln(\alpha+1)} \sum_{n=0}^{\infty} \binom{r}{n} (-1)^{(r-n)} \int_1^\infty \frac{y^n}{(\alpha+1)y^{\theta+1} - \alpha y} dy$$

Let $y = e^u \rightarrow dy = e^u du$, then

$$\grave{\mu}_r = \frac{\lambda^r \alpha\theta}{ln(\alpha+1)} \sum_{n=0}^{\infty} \binom{r}{n} (-1)^{(r-n)} \int_0^\infty \frac{e^{nu}}{(\alpha+1)e^{\theta u} - \alpha} du$$

$$= \frac{\lambda^r \alpha\theta}{(\alpha+1)ln(\alpha+1)} \sum_{n=0}^{\infty} \binom{r}{n} (-1)^{(r-n)} \int_0^\infty \frac{e^{-(\theta-n)u}}{1 - \left[\frac{\alpha}{\alpha+1}\right] e^{-\theta u}} du \tag{11}$$

From formula (11) and using another formula BI (27)(7) (p.336) [14], we get the following

$$\grave{\mu}_r = \frac{\lambda^r \theta}{\ln(\alpha+1)} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{(r-n)} \binom{r}{n} \left(\frac{\alpha}{\alpha+1}\right)^{k+1}}{\theta(1+k) - n} \tag{12}$$

Corollary 1.

The first moment $E(x)$ and second-moment $E(x^2)$ of the ALL distribution at the origin point are given by

$$\boldsymbol{E(x) = \frac{\lambda\theta}{\ln(\alpha+1)} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{(1-n)} \binom{1}{n} \left(\frac{\alpha}{\alpha+1}\right)^{k+1}}{\theta(1+k) - n}} \tag{13}$$

$$\boldsymbol{E(x^2) = \frac{\lambda^2\theta}{\ln(\alpha+1)} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{(2-n)} \binom{2}{n} \left(\frac{\alpha}{\alpha+1}\right)^{k+1}}{\theta(1+k) - n}} \tag{14}$$

After using equations (**13**) and (**14**), we calculate the variance as follows.

$$\text{Var(x)} = \frac{\lambda^2\theta}{\ln(\alpha+1)} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{(2-n)} \binom{2}{n} \left(\frac{\alpha}{\alpha+1}\right)^{k+1}}{\theta(1+k)-n} - \frac{\lambda^2\theta^2}{(\ln(\alpha+1))^2} \left[\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{(1-n)} \binom{1}{n} \left(\frac{\alpha}{\alpha+1}\right)^{k+1}}{\theta(1+k)-n}\right]^2$$

(15)

3.6 The Moment Generating Function

The following theorem states the ALL distribution's moment-generating function (mgf).

Theorem 2. The moment-generating function $M_X(t)$ of the ALL distributing is given by:

$$M_X(t) = \frac{\theta}{\ln(\alpha+1)} \sum_{s=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \binom{s}{n} \frac{\lambda^s t^s \left(\frac{\alpha}{\alpha+1}\right)^{k+1}}{s!\,(\theta(1+k)-n)}$$

Proof: The moment-generating function $M_X(t)$ of the random variable X takin from ALL distribution is given by

$$M_X(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} f(x:\alpha,\lambda,\theta)\,dx \,. \tag{16}$$

Where $f(x:\alpha,\lambda)$ It is the PDF of ALL distribution. By using the Maclaurin series for the function e^{tx} yields

$$e^{tx} = \sum_{s=0}^{\infty} \frac{t^s x^s}{s!} \,. \tag{17}$$

From (16) and (17) we get

$$M_X(t) = \sum_{s=0}^{\infty} \frac{t^s}{s!} \int_0^{\infty} x^s f(x:\alpha,\lambda,\theta)\,dx = \sum_{s=0}^{\infty} \frac{t^s}{s!} \acute{\mu}_s \,. \tag{18}$$

Where $\acute{\mu}_s$ It is the *s-th* moment at the origin point (Theorem 1). Finally, by substituting equation (12) into equation (18), we get

$$M_X(t) \;= \frac{\theta}{\ln(\alpha+1)} \sum_{s=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{(s-n)} t^s \lambda^s \binom{s}{n} \left(\frac{\alpha}{\alpha+1}\right)^{k+1}}{s!\,(\theta(1+k)-n)} \tag{19}$$

3.7 Maximum Likelihood Estimators of the ALL distribution

Now we find the estimator of the three parameters if $x_0, x_1, x_2, \ldots, x_n$ Denote the Likelihood function is provided by a random sample taken from the ALL distributing [15].

$$L(\alpha,\lambda,\theta:x_i) = \prod_{i=1}^{n} \frac{\alpha\theta\left[1+\frac{x}{\lambda}\right]^{-(\theta+1)}}{\lambda ln(\alpha+1)\left[\alpha-\alpha\left[1+\frac{x}{\lambda}\right]^{-\theta}+1\right]}$$
$$= \frac{\alpha^n \theta^n \prod_{i=1}^{n}\left[1+\frac{x_i}{\lambda}\right]^{-(\theta+1)}}{\lambda^n (ln(\alpha+1))^n \prod_{i=1}^{n}\left[\alpha-\alpha\left[1+\frac{x}{\lambda}\right]^{-\theta}+1\right]} \tag{20}$$

The national log-likelihood function is

$$\ln L(\alpha,\lambda,\theta:x_i) = \mathrm{n}\ln\alpha + \mathrm{n}\ln\theta - (\theta+1)\sum_{i=1}^{n} ln\left[1+\frac{x}{\lambda}\right] -$$
$$\mathrm{n}\ln\lambda - \ln(\ln(\alpha+1)) - \sum_{i=1}^{n} ln\left(\alpha-\alpha\left[1+\frac{x}{\lambda}\right]^{-\theta}+1\right) \tag{21}$$

applying the partial derivatives of $\ln L(\alpha,\lambda,\theta:x_i)$ w.r.t. the parameter α , λ, and θ, and setting the result to zero, we get the following equation.

$$\frac{\partial \ln(L(\alpha,\lambda,\theta:x_i))}{\partial\alpha} = \frac{n}{\alpha} - \frac{n}{(\alpha+1)\ln(\alpha+1)} - \sum_{i=1}^{n} \frac{1-\left[1+\frac{x}{\lambda}\right]^{-\theta}}{\left(\alpha-\alpha\left[1+\frac{x}{\lambda}\right]^{-\theta}+1\right)},$$

Additionally, if we set this to zero, then we obtain the following.

$$\frac{n}{\alpha}-\frac{n}{(\alpha+1)\ln(\alpha+1)}-\sum_{i=1}^{n}\frac{1-\left[1+\frac{x}{\lambda}\right]^{-\theta}}{\left(\alpha-\alpha\left[1+\frac{x}{\lambda}\right]^{-\theta}+1\right)}=0 \quad (22)$$

And,

$$\frac{\partial\ln(L(\alpha,\lambda,\theta:x_i))}{\partial\lambda}=\frac{(\theta+1)}{\lambda^2}\sum_{i=1}^{n}\frac{x}{\left[1+\frac{x}{\lambda}\right]}-\frac{\mathrm{n}}{\lambda}+\frac{\alpha\theta}{\lambda^2}\sum_{i=1}^{n}\frac{x\left[1+\frac{x}{\lambda}\right]^{-(\theta+)}}{\alpha-\alpha\left[1+\frac{x}{\lambda}\right]^{-\theta}+1}$$

By setting this equal to zero, we get the following.

$$\frac{(\theta+1)}{\lambda^2}\sum_{i=1}^{n}\frac{x}{1+\frac{x}{\lambda}}-\frac{\mathrm{n}}{\lambda}+\frac{\alpha\theta}{\lambda^2}\sum_{i=1}^{n}\frac{x\left[1+\frac{x}{\lambda}\right]^{-(\theta+)}}{\alpha-\alpha\left[1+\frac{x}{\lambda}\right]^{-\theta}+1}=0, \quad (23)$$

And,

$$\frac{\partial\ln(L(\alpha,\lambda,\theta:x_i))}{\partial\theta}=\frac{n}{\alpha}-\sum_{i=1}^{n}ln\left[1+\frac{x}{\lambda}\right]-\alpha\sum_{i=1}^{n}\frac{\left[1+\frac{x}{\lambda}\right]^{-\theta}ln\left[1+\frac{x}{\lambda}\right]}{\left(\alpha-\alpha\left[1+\frac{x}{\lambda}\right]^{-\theta}+1\right)},$$

$$\boldsymbol{\frac{n}{\alpha}-\sum_{i=1}^{n}ln\left[1+\frac{x}{\lambda}\right]-\alpha\sum_{i=1}^{n}\frac{\left[1+\frac{x}{\lambda}\right]^{-\theta}ln\left[1+\frac{x}{\lambda}\right]}{\left(\alpha-\alpha\left[1+\frac{x}{\lambda}\right]^{-\theta}+1\right)}=0} \quad (24)$$

By solving equations ($\mathbf{22}$), (23), and (24) using the Newton–Raphson method, we obtain MLEs for the three parameters $\boldsymbol{\alpha},\boldsymbol{\lambda}$, and $\boldsymbol{\theta}$.

3.8 Order Statistics of the ALL Distribution

We know that if $\boldsymbol{X_{(1)}},\ldots,\boldsymbol{X_{(n)}}$ Denotes the size n the random sample's order statistic. $\boldsymbol{X1},\ldots,\boldsymbol{Xn}$ from the continuous data population taken from the ALL distributing with CDF $\boldsymbol{F_X(x)}$ and PDF $\boldsymbol{f_X(x)}$ then the p.d.f of $\boldsymbol{j-th}$ order random variable $\boldsymbol{X_{(j)}}$ is given as following

$$\boldsymbol{f_{X_{(j)}}(x)=\frac{n!}{(j-1)!\,(n-j)!}\left(\frac{ln(\alpha-\alpha\left[1+\frac{x}{\lambda}\right]^{-\theta}+1)}{ln(\alpha+1)}\right)^{j-1}\left(1-\frac{ln(\alpha-\alpha\left[1+\frac{x}{\lambda}\right]^{-\theta}+1)}{ln(\alpha+1)}\right)^{n-j}}$$

$$\boldsymbol{\frac{\alpha\theta\left[1+\frac{x}{\lambda}\right]^{-(\theta+1)}}{\lambda ln(\alpha+1)\left[\alpha-\alpha\left[1+\frac{x}{\lambda}\right]^{-\theta}+1\right]}} \quad (\mathbf{25})$$

Also, if $\boldsymbol{X_{(1)}},\ldots,\boldsymbol{X_{(n)}}$ Is the order statistic of a r.s. $\boldsymbol{X1},\ldots,\boldsymbol{Xn}$ taken from the data population with $\boldsymbol{F_X(x)}$ Is the CDF and $\boldsymbol{f_X(x)}$ Is the PDF then the joint PDF (JPDF) of $\boldsymbol{X_{(i)}}$, $\boldsymbol{X_{(j)}}$, $\boldsymbol{1\le i\le j\le n}$ is

$$\boldsymbol{f_{X_{(i)},X_{(j)}}(u,v)=\frac{n!}{(i-1)!\,(j-i-1)!\,(n-j)!}\left(\frac{ln(\alpha-\alpha\left[1+\frac{u}{\lambda}\right]^{-\theta}+1)}{ln(\alpha+1)}\right)^{i-1}}$$

$$\left(\frac{\boldsymbol{ln}(\boldsymbol{\alpha}-\boldsymbol{\alpha}\left[\mathbf{1}+\frac{\boldsymbol{v}}{\boldsymbol{\lambda}}\right]^{-\boldsymbol{\theta}}+\mathbf{1})}{\boldsymbol{ln}(\boldsymbol{\alpha}+\mathbf{1})}-\frac{\boldsymbol{ln}(\boldsymbol{\alpha}-\boldsymbol{\alpha}\left[\mathbf{1}+\frac{\boldsymbol{u}}{\boldsymbol{\lambda}}\right]^{-\boldsymbol{\theta}}+\mathbf{1})}{\boldsymbol{ln}(\boldsymbol{\alpha}+\mathbf{1})}\right)^{\boldsymbol{j-i-1}}$$

$$\left(\mathbf{1}-\frac{\boldsymbol{ln}(\boldsymbol{\alpha}-\boldsymbol{\alpha}\left[\mathbf{1}+\frac{\boldsymbol{v}}{\boldsymbol{\lambda}}\right]^{-\boldsymbol{\theta}}+\mathbf{1})}{\boldsymbol{ln}(\boldsymbol{\alpha}+\mathbf{1})}\right)^{\boldsymbol{n-j}}\frac{\boldsymbol{\alpha\theta}\left[\mathbf{1}+\frac{\boldsymbol{u}}{\boldsymbol{\lambda}}\right]^{-(\boldsymbol{\theta}+\mathbf{1})}}{\boldsymbol{\lambda ln}(\boldsymbol{\alpha}+\mathbf{1})\left[\boldsymbol{\alpha}-\boldsymbol{\alpha}\left[\mathbf{1}+\frac{\boldsymbol{u}}{\boldsymbol{\lambda}}\right]^{-\boldsymbol{\theta}}+\mathbf{1}\right]}\cdot\frac{\boldsymbol{\alpha\theta}\left[\mathbf{1}+\frac{\boldsymbol{v}}{\boldsymbol{\lambda}}\right]^{-(\boldsymbol{\theta}+\mathbf{1})}}{\boldsymbol{\lambda ln}(\boldsymbol{\alpha}+\mathbf{1})\left[\boldsymbol{\alpha}-\boldsymbol{\alpha}\left[\mathbf{1}+\frac{\boldsymbol{v}}{\boldsymbol{\lambda}}\right]^{-\boldsymbol{\theta}}+\mathbf{1}\right]}$$

$$-\infty<\boldsymbol{u}<\boldsymbol{v}<\infty \qquad (26)$$

Therefore, the common pdf function for all random sample order statistics is $\boldsymbol{f}_{\boldsymbol{X}_{(1)},\ldots,\boldsymbol{X}_{(n)}}(\boldsymbol{x}_{1},\ldots,\boldsymbol{x}_{n})$, taken from the ALL distributing, is given by

$$f_{X_{(1)},\ldots,X_{(n)}}(x_{1,\ldots,}x_n)=\begin{cases} n!\dfrac{\alpha^n\theta^n\prod_{i=1}^{n}\left[1+\frac{x_i}{\lambda}\right]^{-(\theta+1)}}{\lambda^n(ln(\alpha+1))^n\prod_{i=1}^{n}\left[\alpha-\alpha\left[1+\frac{x}{\lambda}\right]^{-\theta}+1\right]}, & 0<x_1<\cdots<x_n \\ 0 & otherwise\end{cases} \qquad (27)$$

4. Monte Carlo Simulation

To analyze and interpret the Maximum Likelihood Estimation (MLE) technique outlined in this research segment, we will use a simulation framework to assess the theoretical aspects of the three parameters related to the ALL distribution. This section encompasses an exposition of the Monte Carlo simulation experiment pertinent to the study, mainly focusing on the sample sizes generated when the simulation iterations are set at 1000. And, we will elucidate the results of the simulation tests obtained through MATLAB software [16], employing the "BFGS" algorithm to facilitate the execution of both simulation and estimation techniques. The technique of the creation of the simulation experiments integrates a multitude of key elements, particularly the determination of sample sizes, which were as follows (n = 100, 300, 500, 700, 900), along with the choice of parameter values for five individual experiments, where the parameter values are $(\alpha,\ \lambda,\ \theta)$=(1,5, 1.5, 0.5), $(\alpha,\ \lambda,\ \theta)$=(1.5, 3, 1.5), $(\alpha,\ \lambda,\ \theta)$=(3, 2, 4), $(\alpha,\ \lambda,\ \theta)$=(0.3, 5, 2.2) and $(\alpha,\ \lambda,\ \theta)$=(1.8, 7, 1.2). Additionally, this process involves the generation of suitable data for the ALL distribution and the computation of the MLE for the three parameters $(\alpha,\ \lambda,\ \theta)$, followed by the assessment of mean square errors (MSE) and biases (Bias). The outputs from the simulation are subsequently demonstrated in Table 1 in Figure 5 to mean square errors and Figure 6 to biases.

Table 1**:** The MSEs and Biases of the $(\hat{\alpha},\hat{\lambda},\hat{\theta})$ to different sample sizes

Par.	Size	MSE(α)	Bias(α)	MSE(λ)	Bias(λ)	MSE(θ)	Bias(θ)
$\alpha=1.5$	100	35.0723	4.5188	6.8648	2.1297	32.0362	4.5927
	300	41.4106	5.6745	4.8662	1.961	30.418	4.8516
$\lambda=1.5$	500	40.581	5.6936	3.8999	1.7878	24.4284	4.3887
$\theta=1.5$	700	45.1189	6.3583	4.6382	1.9906	30.2298	5.0817
	900	45.9866	6.4406	4.1784	1.9442	27.9244	4.9825
$\alpha=1.5$	100	27.2442	3.458	21.9753	3.2135	17.611	2.9255
	300	23.235	3.7275	12.6493	2.4256	12.2753	2.4814
$\lambda=3$	500	18.1789	3.3944	11.2542	2.2031	11.6787	2.3074
$\theta=1.5$	700	16.3188	3.3335	5.9545	1.6162	6.3478	1.7427
	900	15.2595	3.2821	6.2845	1.6472	6.4862	1.7934
	100	16.0903	2.1624	4.3504	1.8765	28.5507	4.8422
$\alpha=3$	300	14.0443	2.9067	3.8501	1.8863	32.9374	5.5777
	500	8.7247	2.3955	4.1633	2.0032	34.0047	5.7582

$\lambda = 2$ $\theta = 4$	700	11.0084	3.0182	3.8456	1.943	35.9999	6
	900	8.0499	2.5427	4.1258	2.0169	35.4691	5.9459
$\alpha = 0.3$ $\lambda = 5$ $\theta = 2.2$	100	8.7017	1.7585	17.0681	3.1128	10.7041	2.4344
	300	2.6078	1.0563	11.7886	2.0924	7.1816	1.7201
	500	2.8988	1.254	11.9717	2.36	7.1782	1.9152
	700	2.8475	1.3326	12.1003	2.6064	7.2384	2.0737
	900	2.3447	1.2373	9.7737	2.2131	5.7753	1.7964
$\alpha = 1.8$ $\lambda = 7$ $\theta = 1.2$	100	14.4835	2.3887	8.4974	1.7649	0.59918	0.46179
	300	10.2248	2.5582	7.5919	1.6959	0.53752	0.50035
	500	10.6451	2.898	6.4182	1.4784	0.49677	0.49264
	700	11.1754	3.055	6.7851	1.8906	0.55735	0.59595
	900	10.154	2.9306	6.4942	1.8398	0.51539	0.56846

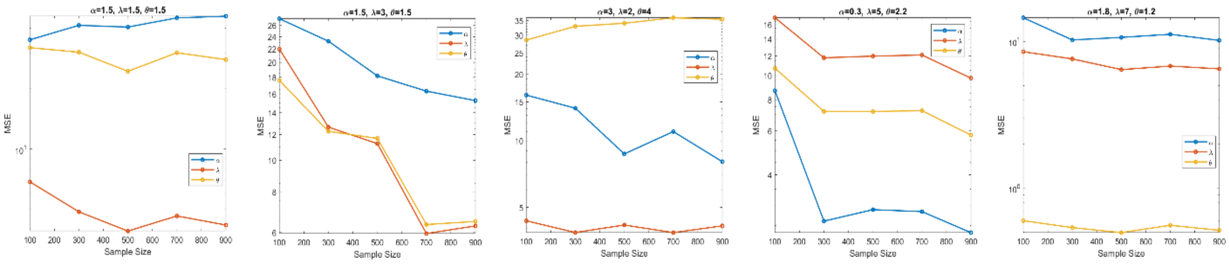

Figure 5. The MSE Plots the ALL distribution MLEs

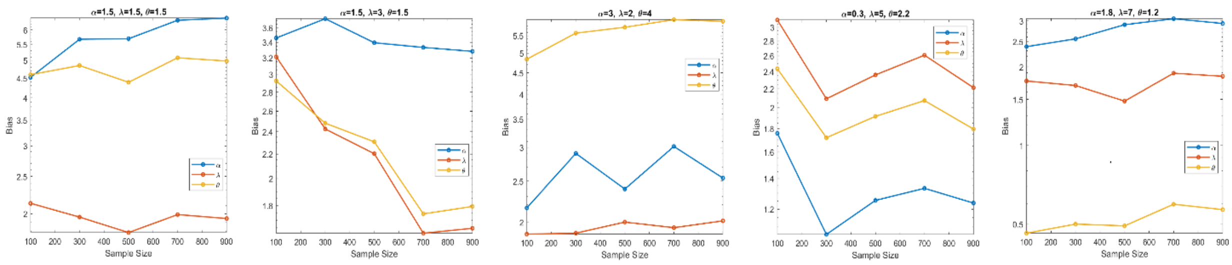

Figure 6. The Baes Plots the ALL distribution MLEs

5. Application

In this section, we delineate a practical application to exemplify the significance of the ALL distribution. This dataset encapsulates the duration of waiting times (measured in minutes) before receiving service for a cohort of 100 bank clients and was thoroughly scrutinized and analyzed by Ghitany et al. (2013) [17]. This dada set is "0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 3.6, 4.0, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9, 5.0, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8.0, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7, 10.9, 11.0, 11.0, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 12.9, 13.0, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19.0, 19.9, 20.6, 21.3, 21.4, 21.9, 23.0, 27, 31.6, 33.1, 38.5".

We were using this data set to compare the performance of the proposed ALL distribution and some of the other distributions, namely the Lumex Gompertz distribution (LGD), the Transformed Gompertz distribution (TGD), the odd generalized exponential Gompertz distribution (OGEGD), the Generalized Gompertz distribution (GGD), the Gompertz distribution (GD), Transmuted Lomax distribution (TLD), Gamma Lomax distribution (GaLD), Beta Lomax distribution (BLD), Marshall-Olkin Lomax distribution (MOLD) and Lomax distribution (LD).

Table 2: Results of comparing the ALL distribution with other distributions for the selected data.

Distributions	L	AIC	CAIC	BIC	HQIC
LGD	347.8476	703.6952	704.1162	714.1159	707.9126
TGD	365.8488	737.6975	737.9475	745.5130	740.8606
OGEGD	659.982	1327.965	1328.387	1338.386	1332.183
GGD	739.504	1485.009	1485.259	1492.824	1488.172
GD	2894.288	5792.575	5792.699	5797.785	5794.684
TLD	366.281	738.5721	738.833	746.387	741.735
GaLD	369.391	767.604	768.242	780.630	772.875
BLD	368.088	743.588	744.009	754.008	747.805
MOLD	360.092	730.184	730.882	743.210	735.456
LD	409.154	822.308	822.432	827.518	824.417
ALLD	322.673	651.346	662.162	659.162	654.509

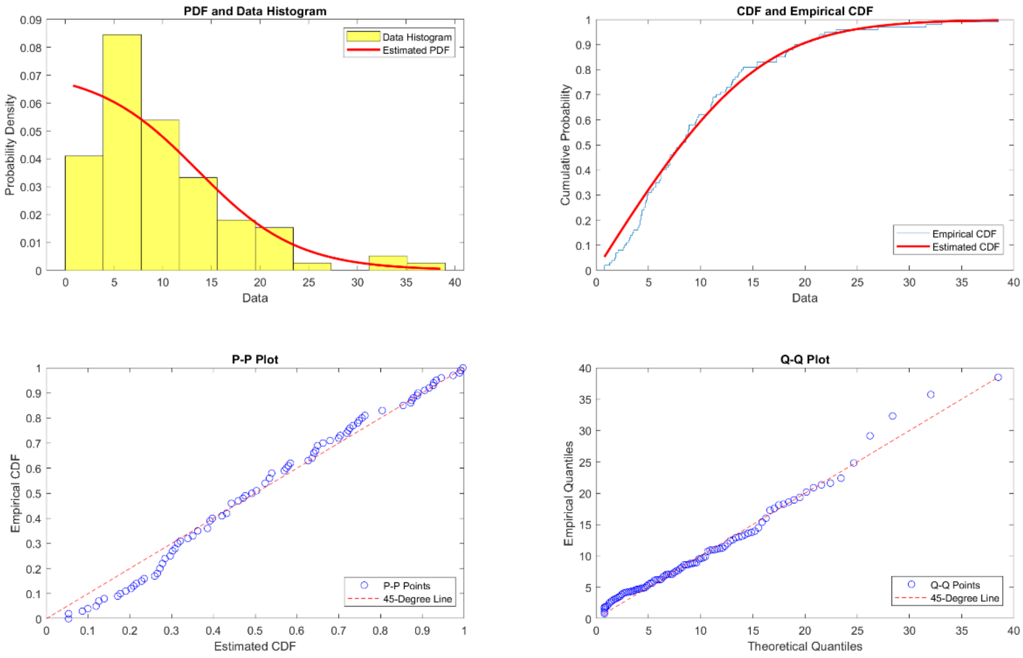

Figure 7. Test results from the selected data with the ALL distribution.

6. Conclusions

This paper introduces the continuous Alpha logarithm Lomax (ALL) distribution with three parameters through the selection of the Lomax distribution, which is regarded as proper for the ALG distributions family, entailing the addition of a third parameter by using the Alpha logarithm-G (AL-G) family method with the Lomax distribution. Statistical functions and qualities of the ALL model were examined. The validity of this model was corroborated by using a real dataset, confirming it's superior for the selected data compared to the other distributions.

References

[1] Lomax, Kenneth S. The Sources and Nature of Statistical Information in Special Fields of Statistics Regional Economic Statistics. Journal of the Royal Statistical Society. Series A (General), 1954, 117.1: 85-101.

[2] R., A., Bakoban., Eman, A., Faraj., Najwa, S., Alsulami. A study on alpha power Lomax distribution. International journal of engineering science technologies, (2022).;6(5):76-91. doi: 10.29121/ijoest.v6.i5.2022.393

[3] Mustafa, Kamal., Ramy, Aldallal., Said, G., Nassr., Aned, Al, Mutairi., Musa, Jimoh, Yusuf., Manahil, Mustafa., M., Alsolmi., Ehab, M., Almetwally. A new, improved form of the Lomax model: Its bivariate extension and an application in the financial sector. Alexandria Engineering Journal, (2023).;75:127-138. Doi: 10.1016/j.aej.2023.05.027

[4] Lal, Babu, Sah, Telee., Ram, Suresh, Yadav., Vijay, Kumar. Modified Lomax Distribution: Properties and Applications. (2024). doi: 10.21203/rs.3.rs-4265580/v1

[5] Mahreen, Abdullah., Muhammad, Ahsan-ul-Haq., Abdullah, Alomair., Mohammed, Alomair. Classical and Bayesian inference for the new four-parameter Lomax distribution with applications. Heliyon, (2024). doi: 10.1016/j.heliyon.2024.e25842

[6] L.A. Mohsin and H.G. Kalt,“ An application to the exponential distribution of a novel family of distributions,” AIP Conference Proceedings, in press.

[7] Kalt HG. A New Family of Continuous Distributions with Applications. Stat., optim. Inf. Comput. [Internet]. 2024Aug.23 [cited 2024Sep.28];. Available from: http://www.iapress.org/index.php/soic/article/view/2144

[8] Khadim A., Saghir A., Hussain T., & Shakil M.**,** Some New Developments and Review on T-X Family of Distributions, Journal of Statistics Applications & Probability, J. Stat. Appl. Pro. 11, No. 3, 739-757 (2022).

[9] H.G. Kalt and I.H. Hussein,“ Bayesian Estimation for Parameters and the Survival Function of Modified Weibull Distribution using Tierney and Kadane’s Approximation,” Jour of Adv Research in Dynamical and Control Systems, vol. 10, 10-Special Issue, pp. 1960 –1070, 2018.

[10] Alzaatreh, A., Lee, C., & Famoye, F. (2013). A new method for generating families of continuous distributions. Metron, 71(1), 63-79.

[11] H.G. Kalt and I.H. Hussein,“ A comparison of MLE method and OLSE for the estimation of modified Weibull distribution parameters by using the simulation,” ARPN Journal of Engineering and Applied Sciences, vol. 13, no. 17, pp. 4833 –4839, 2018.

[12] Hassan, A. S., & Elgarhy, M. (2016). Kumaraswamy Weibull-generated family of distributions with applications. Advances and Applications in Statistics, 48(3), 205.

[13] N.J. Hassan, H.G. Kalt, H.A. Aal-Rkhais and A.E. Hashoosh,“ The Weibull Lindley: General family of probability distributions,” Mathematics in Engineering, Science and Aerospace (MESA), vol. 12, no. 1, pp. 29–42, 2021

[14] I. S. Gradshteyn and I. M. Ryzhik,” Table of Integrals, Series, and Products” Eighth Edition. Tulane University, Department of Mathematics.

[15] Cordeiro, G. M., & de Castro, M. (2011). A new family of generalized distributions. Journal of statistical computation and simulation, 81(7), 883-898.

[16] H.G. Kalt and MK Abd Al-Rahem,“ Comparison of four methods to estimate the scale parameter for Extended Poisson Exponential distribution,” in IOP Conference Series: Materials Science and Engineering, vol. 571, no. 1, pp. 012029, 2019.

[17] O. Aisha “A STUDY ON THE PROPERTIES AND APPLICATIONS OF LOMAX-GOMPERTZ DISTRIBUTION.” DEPARTMENT OF STATISTICS, FACULTY OF PHYSICAL SCIENCES AHMADU BELLO UNIVERSITY, ZARIA, NIGERIA (2018).