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Petrov Weak Galerkin Finite Element Method for Nonlinear Solving Convection Diffusion Problem

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Abstract:

In this paper, described and analysis, For the nonlinear Convection-Diffusion issue, the Petrov Weak Galerkin finite element (PWGFE) approach established that the bilinear form of $a_{PW}(u,v)$ is ψ_r −elliptic. The semi-discrete stability and error estimate are demonstrated, and the approximations have errors of $O(h^2), O(h+\tau)$, respectively. The convergence and correctness of the PWGFE approach are confirmed by the numerical findings.

Keywords: Weak Galerkin Finite Element, Nonlinear Convection Diffusion equation, PWG Finite Element method, the Stability, the error analysis.

1.Introduction:

In this paper, we take into account the Dirichlet issue for time-tributary a nonlinear convection diffusion equation that aim to find an unidentified function.
$u = u(x,t)$ satisfying:

$$u_t - \epsilon\,\Delta u + u \cdot \nabla u + cu = f, \qquad (x,t) \in \Omega \times (0,T], \tag{1.1}$$

$$u = 0, \qquad (x,t) \in \Gamma \times (0,T], \tag{1.2}$$

$$u(.,0)|\Gamma = g, \qquad x \in \Omega, \tag{1.3}$$

where Ω is polygonal in R^2 with the boundary $\Gamma = \partial\Omega, u_t = \frac{\partial u}{\partial t}$, and $f,\ g$ are given functions; also, $\epsilon > 0$ is a diffusion coefficient; and ∇u indicates the gradient of the function $u = u(x,t)$.

Since the 1950s, researchers include developed time-stepping techniques to roughly numerically estimate solutions to these issues. There are two groups of numerical approaches for such nonlinear convection-diffusion situations. The first group of finite difference algorithms replaces the differential quotient with a quotient., see [1,2] and the more refers to as FEMs, see [3,4]. In physics, mechanics, and biology, nonlinear convection-diffusion problems are frequently exercised to describe a wide variety of processes and phenomena. In order to solve a second order elliptic issue and subsequent partial differential equations [6–14], Wang and Ye [5] recently developed WG FEM.

This paper orderly as follows: - In section 2, show Petrov weak Galerkin space and the condition to the stability parameter. In Section 3, we demonstrate a modern weak variation for (1.1) -(1.3) create nonlinear problem. In section 4, solve nonlinear convection diffusion by Petrov Weak Galerkin method .In section 5, we prove important property is $\psi_r - elliptic$ and the stability of the Petrov Weak Galerkin finite element method is analysis. In section 6, the error analysis semi-discrete PWGFE method. In section 7, To support the theoretical findings, provide some numerical results. Finally, In section 10, discussion and conclusion

2. Petrov Weak Galerkin Spaces:

When a test function space differs from the trial function space, Petrov Weak Galerkin mothed is used. PWGFE mothed was defined in this section. Let $\{T_h\}$ represent the family of triangulation shapes of which $0 < h < 1$ represents. $P_j(T^0)$ stands for the set of polynomials on T^0 with a maximum degree of j, and $P_\ell(\partial \mathrm{T})$ stands for the set of polynomials on ∂T with a maximum degree of ℓ. A weak function $v = \{v^0, v^b\}$ such that $v^0 \in P_j(T^0)$ and $v^b \in P_\ell(\partial T)$ with $j \geq 0$ and $\ell \geq 0$ is referred to as a discrete weak function $v = \{v^0, v^b\}$.

The discrete weak function space indicated by $W(T, j, \ell)$ is

$$W(T, j, \ell) \coloneqq \{v = \{v^0, v^b\}: v^0 \in P_j(T^0), v^b \in P_\ell(\partial T) \cdot \qquad (2.1)$$

Patching across all of the triangles $T \in T_h$ would define the corresponding finite element space. Alternatively said, the weak finite element space is defined as

$$S_h(j, \ell) = \{v = v^0, v^b\}: \{v^0, v^b\} \mid_T \in W(T, j, \ell), \forall\, T \in T_h\} \quad . \qquad (2.2)$$

The $S_h(j, \ell)$'s subspace with vanishing boundary values on $\partial\Omega$ is denoted as $S_h^0(j, \ell)$. i.e.,

$$S_h^0(j, \ell) = \{v = \{v^0, v^b\} \in S_h(j, \ell), v^b|_{\partial T \cap \partial \Omega} = 0 \quad , \forall\, T \in T_h\} \ . \qquad (2.3)$$

Let ψ_r be trail space and ϕ_r be a test space which are define

$$\psi_r = \{v = \{v^0, v^b\} \in W(T, j, \ell), \forall T \in \ T_h\}$$

$$\phi_r = \{\ \omega : \omega = v\text{^}0 + \delta\, \beta \cdot \nabla\text{\_}(d, r\ )\ \ v\ ;\ v\ \in \psi\text{\_}r\ \}$$

and

$$\psi_r^0 = \{v = \{v^0, v^b\} \in \psi_r, v^b|_{\partial T\, \cap \partial \Omega} = 0, \forall T \in \ T_h\}$$

$$\phi_r^0 = \left\{\omega : \omega = v^0 + \delta\, \beta \cdot \nabla_{d,r}\ v\ ;\ v\ \in \psi_r^0\right\} \qquad (2.4)$$

In this context, " δ " symbolize a constant stability parameter. It'll be designated as [15];

$$\delta = \begin{cases} \zeta\, h & if \quad \epsilon\ < h \\ 0 & if \quad \epsilon \geq h \end{cases} \qquad ;\ 0 < \zeta < \frac{1}{4}\ , (\text{small constant}\ ) \cdot$$

and $dim\, \psi_r = dim\, \phi_r$.

3.Weak variational form

We first establish a unique weak variational form connected to issue (1.1) - (1.3) in order to describe the WG approximation to problems (1.1) and (1.3) using finite elements. Multiplying equation (1.1) by test function $v \in H_0^1(\Omega)$ and integrating together sides on Ω , we have.

$$(u_t\, , v) + (\epsilon \nabla u, \nabla v) + (u \cdot \nabla u, v) + (cu, v) = (f, v) \qquad (3.1)$$

Rewritten term (the third) in (3.1) we get,

$$(u \cdot \nabla u, v) = \frac{1}{3}\ (u \cdot \nabla u, v) + \frac{1}{3}\ (\nabla(\ u^2, v), \qquad (3.2)$$

and term (the second) in the aforementioned equation, integrating by part, we get

$$(u \cdot \nabla u, v) = \frac{1}{3}\ (u \cdot \nabla u, v) - \frac{1}{3}\ (\nabla(uu\ , v), \qquad (3.3)$$

Substituting (3.3) in to equation (3.1), the weak form for equation (3.1) is to find $u \in H^1(0, \mathrm{T}; H_0^1(\Omega))$, for each $v \in H_0^1(\Omega)$, such that

$$(u_t\ , v) + (\epsilon \nabla u, \nabla v) + \frac{1}{3}\ (u \cdot \nabla u, v) - \frac{1}{3}\ (\nabla(uu\ , v)$$

$$+(cu, v) = (f, v)\ ,\ x \in\ \Omega, \qquad (3.4)$$

$$u(x, 0)\ =\ g,\ x \in\ \Omega, \qquad (3.5)$$

4. Petrov Weak Galerkin Finite Element Method:

The WG finite element technique, which is to determine $u = \{u^0, u^b\} \in S_h(j, \ell)$, satisfies $u^b = Q^b g$ on $\partial\Omega$ such that [12] is based on (1.1)-(1.3) and (3.3).

$$(u_t, v^0) + a_W(u_h, v) = (f, v^0), \quad \forall v = \{v^0, v^b\} \in S_h^0(j, \ell) \qquad (4.1)$$

Where

$$a_W(u_h; u_h, v) = a_0(u_h, v) + a_1(u_h; u_h\ , v) \qquad (4.2)$$

ie.,

$$a_0(u_h, v)\ = \left(\epsilon \nabla_{d,r} u\ , \nabla_{d,r} v\right) + \left(cu_h\ , v^0\ \right) \qquad (4.3)$$

$$a_1(u_h; u_h, v) =\ \frac{1}{3}\ \left(u_h^0 \cdot \nabla_{d,r} u\ , v^0\right) - \frac{1}{3}\ (\left(u_h^0\, u_h^0, \nabla_{d,r} v\right) \qquad (4.4)$$

In equations (4.1) - (4.4) we change v byω , then the semi –discrete PWG finite element method is:
Find $u_h = \{u_h^0, u_h^b\} \in \psi_r$ such that

$$\left(u_{h,t}\ , v^0\right) + \left(u_{h,t}, \delta\beta \cdot \nabla_{d,r} v\ \right) + \left(\epsilon \nabla_{d,r} u_h, \nabla_{d,r} v\right) + (cu_h, v^0) +$$

$$\left(cu_h, \delta\, \beta \cdot \nabla_{d,r} v\right) +\ \ \frac{1}{3}\ \left(u_h^0 \cdot \nabla_{d,r} u\ , v^0\right) + \frac{1}{3}\ \left(u_h^0 \cdot \nabla_{d,r} u\ , \delta\, \beta \cdot \nabla_{d,r} v\right)$$

$$-\frac{1}{3}\ (\left(u_h^0\, u_h^0, \nabla_{d,r} v\right) = (f, v^0) + \left(f, \delta\, \beta \cdot \nabla_{d,r} v\right), \quad \forall v =\ \{v^0, v^b\} \in \psi_r^0\ , \qquad (4.5)$$

or

$$\left(u_{h,t}\ , v^0\right) + \left(u_{h,t}, \delta\beta \cdot \nabla_{d,r} v\ \right) + a_{PW}(u_h; u_h, v) =$$

$$(f, v^0) + \left(f, \delta\, \beta \cdot \nabla_{d,r} v\right),\ \forall v =\ \{v^0, v^b\} \in \psi_r^0$$

where,

$$a_{PW}(u_h; u_h, v) = \left(\epsilon\nabla_{d,r}u_h, \nabla_{d,r}v\right) + (cu_h, v^0) + \left(cu_h, \delta\,\beta\cdot\nabla_{d,r}v\right)$$
$$+\frac{1}{3}\ \left(u_h^0\cdot\nabla_{d,r}u\,, v^0\right) + \frac{1}{3}\ \left(u_h^0\cdot\nabla_{d,r}u\,, \delta\,\beta\cdot\nabla_{d,r}v\right) \ -\frac{1}{3}\ (\left(u_h^0\,u_h^0, \nabla_{d,r}v\right)$$
$$\forall v = \{v^0, v^b\} \in \psi_r^0\,, \qquad (4.6)$$

Moreover, the fully discrete formulation is as follows:

Find $u_h^n \in \psi_r$ such that

$$\left(\bar{\partial}u_h^n\,, v^0\right) + \left(\bar{\partial}u_h^n, \delta\,\beta\cdot\nabla_{d,r}\ v\right) + \left(\epsilon\nabla_{d,r}u_h^n\,, \nabla_{d,r}v\right)$$
$$+(cu_h^n, v^0) + \left(cu\,_h^n, \delta\,\beta\cdot\nabla_{d,r}v\right) + \frac{1}{3}\ \left(u_h^0\cdot\nabla_{d,r}u\,, v^0\right)$$
$$+\frac{1}{3}\ \left(u_h^0\cdot\nabla_{d,r}u\,, \delta\,\beta\cdot\nabla_{d,r}v\right) - \frac{1}{3}\ (\left(u_h^0\,u_h^0, \nabla_{d,r}v\right)$$
$$= (f^n, v^0\,) + \left(f^n, \delta\,\beta\cdot\nabla_{d,r}v\right), \forall v = \{v^0, v^b\} \in \psi_r^0 \quad , \qquad (4.7)$$

or,

$$\left(\bar{\partial}u_h^n\,, v^0\right) + \left(\bar{\partial}u_h^n, \delta\,\beta\cdot\nabla_{d,r}\ v\right) + \ a_{PW}(u_h; u_h, v)$$
$$= (f^n, v^0\,) + \left(f^n, \delta\,\beta\cdot\nabla_{d,r}v\right), \forall v = \{v^0, v^b\} \in \psi_r^0$$

where,

$$\bar{\partial}u_h^n = \left(\frac{u_h^n - u_h^{n-1}}{\tau}\right),$$

$$a_{PW}(u_h; u_h, v) = \left(\epsilon\nabla_{d,r}u_h^n\,, \nabla_{d,r}v\right) + (cu_h^n, v^0) + \left(cu\,_h^n, \delta\,\beta\cdot\nabla_{d,r}v\right)$$
$$+\ \frac{1}{3}\ \left(u_h^0\cdot\nabla_{d,r}u\,, v^0\right) + \frac{1}{3}\ \left(u_h^0\cdot\nabla_{d,r}u\,, \delta\,\beta\cdot\nabla_{d,r}v\right) \ -\frac{1}{3}\ (\left(u_h^0\,u_h^0, \nabla_{d,r}v\right), \qquad (4.8)$$

5: Stability for PWGFEM

In this section, we demonstrated the PWGFE method's stability using equation (4.5). that allow the problem data to be used to estimate a suitable norm for a solution.

Lemma $\mathbf{5.1}$: $(\psi_r - \boldsymbol{elliptic})$:

Let $a_{Pw}(\cdot\ ,\cdot)$ be the bilinear form which define in equation (4.6). If

$$0 < \delta \leq \frac{\xi}{2\|c\|_{L\infty}^2} \quad . \qquad (5.1)$$

Then,

$$a_{PW}(v_h; v_h, v_h) \geq \frac{1}{2}\ \|v_h\|_{PW}^2, \qquad (5.2)$$

where,

$$\|u_h\|_{PW} = \ (\epsilon \left\|\nabla_{d,r} u_h\right\|^2 + \|c\|_{L_\infty} \|u^0\|^2 + \xi \left\|\beta \cdot \nabla_{d,r} u_h\right\|^2 + \frac{1}{3} \left\|u^0 . \nabla_{d,r} u_h\right\|^2)^{\frac{1}{2}}, \qquad (5.3)$$

Proof:

Let $u = v^0$ in equation (5.3), we get,

$$a_{PW}(v_h; v_h, v_h) = \ \left(\epsilon \nabla_{d,r} v\,, \nabla_{d,r} v\right) + (c v^0, v^0) + \left(c v^0, \delta\, \beta \cdot \nabla_{d,r} v\right) + \frac{1}{3}\ \left(v^0 \cdot \nabla_{d,r} v, v^0\right) + \frac{1}{3}\ \left(v^0 \cdot \nabla_{d,r} v, \delta\, \beta \cdot \nabla_{d,r} v\right) - \frac{1}{3}\ \left(v^0\ v^0, \nabla_{d,r} v\right), \qquad (5.4)$$

From [8], we have

$$\frac{1}{3}\ \left(v^0 \cdot \nabla_{d,r} v, v^0\right) = \frac{1}{3}\ \left(v^0\ v^0, \nabla_{d,r} v\right), \qquad (5.5)$$

we obtain;

$$a_{PW}(v_h; v_h, v_h) = \ \|\epsilon\|_{L_\infty} \left\|\nabla_{d,r} v\right\|^2 + \|c\|_{L_\infty} \|v^0\|^2 + \left(c v^0, \delta\beta \cdot \nabla_{d,r} v\right) + \ \frac{1}{3}\ \left(v^0 \cdot \nabla_{d,r} v\,, \delta\, \beta \cdot \nabla_{d,r} v\right), \qquad (5.6)$$

In term third of equation (5.6), we get;

$$\left|\left(c v^0, \delta\beta \cdot \nabla_{d,r} v\right)\right| \leq \ \delta^{\frac{1}{2}} \|c\|_{L_\infty}\ \|v^0\|\ \delta^{\frac{1}{2}}\ \|\beta\ \cdot \nabla_{d,r} v\|, \qquad (5.7)$$

From the provision equation (5.1) , we get;

$$\left|\left(c v^0, \delta\beta \cdot \nabla_{d,r} v\right)\right| \leq \frac{\sqrt{\xi}}{\sqrt{2}\ \|c\|_{L_\infty}}\ \|c\|_{L_\infty}\ \|v^0\|\ \delta^{\frac{1}{2}} \|\beta\ \cdot \nabla_{d,r} v\| \leq \frac{\sqrt{\xi}}{\sqrt{2}}\ \|v^0\|\ \ \delta^{\frac{1}{2}} \|\beta\ \cdot \nabla_{d,r} v\|, \qquad (5.8)$$

In term four of equation $(5.6))$, by Cauchy – Schwartz inequality, we have;

$$\frac{1}{3} \left|\left(v^0 \cdot \nabla_{d,r} v, \delta\beta \cdot \nabla_{d,r} v\right)\right| \leq \frac{1}{3} \left(\delta^{\frac{1}{2}} \|c\|_{L_\infty} \left\|v^0 . \nabla_{d,r} v\right\|\ \delta^{\frac{1}{2}}\ \left\|\beta\ \cdot \nabla_{d,r} v\right\|\right), \qquad (5.9)$$

by Young's - inequality, we obtain;

$$\frac{1}{3}\left|\left(v^{0}\cdot\nabla_{d,r}v\,,\delta\beta\cdot\nabla_{d,r}v\right)\right|\leq\xi\left\|v^{0}\cdot\nabla_{d,r}v\right\|^{2}+\frac{1}{12}\,\delta\left\|\beta\;\cdot\nabla_{d,r}v\right\|^{2},\qquad(5.10)$$

Where, $(\frac{1}{2\sigma}(\ \delta)) = \xi$, let $\sigma = 1$

From equation (5.8), equation (5.10) and using Young's - inequality, we have;

$$\left|\left(cv^{0},\delta\beta\cdot\nabla_{d,r}v\right)\right|\leq\frac{\xi}{2}\,\|v^{0}\|^{2}+\frac{1}{4}\,\delta\left\|\beta\;\cdot\nabla_{d,r}v\right\|^{2}$$

$$\leq\frac{1}{2}\|v_{h}\|_{Pw}^{2}\,.\qquad(5.11)$$

and,

$$\left|\left(v^{0}\cdot\nabla_{d,r}v\,,\delta\beta\cdot\nabla_{d,r}v\right)\right|\leq\frac{1}{2}\|v_{h}\|_{Pw}^{2}\,.\qquad(5.12)$$

By used definition $\|v_h\|_{PW}$ and [16], we get;

$$a_{PW}(v_{h};v_{h}\,,v_{h}\,)\geq\|v_{h}\|_{Pw}^{2}\;-\left|\left(cv^{0},\delta\beta\cdot\nabla_{d,r}v\right)\right|.\qquad(5.13)$$

From equation (5.13)

$$a_{PW}(v_{h};v_{h}\,,v_{h}\;)\geq\|v_{h}\|_{Pw}^{2}-\frac{1}{2}\,\|v_{h}\|_{Pw}^{2}\;,\qquad(5.14)$$

$$a_{PW}(v_{h};v_{h}\,,v_{h}\;)\geq\frac{1}{2}\,\|v_{h}\|_{Pw}^{2}\;.\qquad(5.15)$$

The proof is complete. ■

Theorem 5.1:

Let Lemma 5.1 be achieved and the solution of (4.5) satisfies, there exist a constant $\gamma > 0$ dependent on ξ , δ such that

$$\left\|u_{h}\,(x,t)\,\right\|^{2}-\;\|u_{h}(x,0)\,\|^{2}+\int_{0}^{t}\left\|u_{h}\,\right\|_{Pw}^{2}dt$$

$$\leq\int_{0}^{t}2\,\gamma\,\|f(x,t)\|^{2}dt+\int_{0}^{t}\left(\frac{\xi}{2}\,\left\|u_{h}\,\right\|^{2}-2\left\|\,u_{h,t}\right\|^{2}\right)dt\qquad(5.16)$$

Proof : Let $v = u_h$ in equation (4.5) we get ;

$$\left(u_{h,t}\,,u_{h}\right)+\left(u_{h,t}\,,\delta\beta\cdot\nabla_{d,r}u_{h}\,\right)+a_{PW}(u_{h}\colon u_{h}\,,u_{h}\;)$$

$$=\left(f,u_{h}\,\right)+\;\left(f,\delta\,\beta\cdot\nabla_{d,r}u_{\;h}\right)\qquad.\qquad(5.17)$$

If we take the terms (from the left side) of equation 5.17), we get;

$$\left(u_{h,t},u_{h}\right)=\frac{1}{2}\,\frac{d}{dt}\,\|u_{h}(t)\,\|^{2}\qquad,\qquad(5.18)$$

we get using the Cauchy-Schwartz inequality and the Young's inequality;

$$\left(u_{h,t}, \delta\beta \cdot \nabla_{d,r} u_{h}\right) \leq \left\|u_{h,t}\right\| \left\|\delta\beta \cdot \nabla_{d,r} u_h\right\|$$

$$\leq \frac{\vartheta}{2} \left\|u_{h,t}\right\|^2 + \frac{1}{2\vartheta}\, \delta \left\|\beta \cdot \nabla_{d,r} u_h\right\|^2, \qquad (5.19)$$

Chose $\vartheta = 2$, we have;

$$\leq \left\|u_{h,t}\right\|^2 + \frac{1}{4}\, \delta \left\|\beta \cdot \nabla_{d,r} u_h\right\|^2 \quad . \qquad (5.20)$$

From Lemma 5.1, we have;

$$a_{PW}(u_h; u_h, u_h) \geq \frac{1}{2} \|u_h\|_{Pw}^2 \quad . \qquad (5.21)$$

gather the first term in the (right side), then using the Cauchy-Schwartz and Young's inequality, we may derive;

$$(f, u_h) \leq \frac{1}{\sqrt{\xi}} \|f\| \sqrt{\xi}\, \|u_h\|$$

$$\leq \frac{1}{\mu} \|f\|^2 + \frac{\xi}{4} \|u_h\|^2 \ . \qquad (5.22)$$

gather the second term in (right side), then using the Cauchy-Schwartz and Young's inequality, we may derive;

$$\left(f, \delta\,\beta \cdot \nabla_{d,r} u_{h}\right) \leq \left\|\delta^{\frac{1}{2}} f\right\| \left\|\delta^{\frac{1}{2}}(\beta \cdot \nabla_{d,r} u_h)\right\| , \qquad (5.23)$$

Young's inequality applied to equation (5.23), we get;

$$\leq \delta \|f\|^2 + \frac{1}{4}\, \delta \left\|\beta \cdot \nabla_{d,r} u_h\right\|^2 . \qquad (5.24)$$

Substituting (5.20), (5.21), (5.22), (5.23) and (5.24) in (5.17)

we have;

$$\frac{1}{2}\frac{d}{dt} \|u_h(t)\|^2 + \left\|u_{h,t}\right\|^2 + \frac{1}{4}\, \delta \left\|\beta \cdot \nabla_{d,r} u_h\right\|^2 + \frac{1}{2} \|u_h\|_{Pw}^2$$

$$\leq \frac{1}{\xi} \|f\|^2 + \frac{\xi}{4} \|u_h\|^2 + \delta \|f\|^2 + \frac{1}{4}\, \delta \left\|\beta \cdot \nabla_{d,r} u_h\right\|^2 .$$

This implies

$$\frac{1}{2}\frac{d}{dt} \|u_h(t)\|^2 + \frac{1}{2} \|u_h\|_{Pw}^2 \leq \gamma \|f\|^2 + \frac{\xi}{4} \|u_h\|^2 - \left\|u_{h,t}\right\|^2, \qquad (5.25)$$

where $\gamma = (\frac{1}{\xi} + \delta)$, multiply equation (5.25) by 2 and taking the integral on together sides from (0 to t),

we get;

$$\|u_h(x,t)\|^2 - \|u_h(x,0)\|^2 + \int_0^t \|u_h\|_{Pw}^2 dt \leq \int_0^t 2\gamma \|f(x,t)\|^2 dt + \int_0^t \left(\frac{\xi}{2}\|u_h\|^2 - 2\|u_{h,t}\|^2\right) dt. \quad (5.26)$$

Led to

$$\|u_h(x,t)\|^2 + \int_0^t \|u_h\|_{Pw}^2 dt - \int_0^t \left(\frac{\xi}{2}\|u_h\|^2 - 2\|u_{h,t}\|^2\right) dt \leq \int_0^t 2\gamma \|f(x,t)\|^2 dt - \|u_h(x,0)\|^2, \quad (5.27)$$

The proof is complete. ∎

6. The Error Analysis of Semi-discrete PWGFE method

In this subsection, We generate error estimates in the L^2 − norm for the semi-discrete PWGFE approach.

Lemma 6 · 1 [17]: For $u \in H^{1+r}(\Omega)$ with $r > 0$,we have ;

$$\|\pi_h(\epsilon\nabla u) - \epsilon R_h(\nabla u)\| \leq Ch^r \|u\|_{1+r} \cdot \quad (6.1)$$

Lemma 6 · 2 [18]: For $u \in H^{1+r}(\Omega)$ with $r > 0$,we have;

$$\|u - \pi_h u\| \leq Ch^r \|u\|_{1+r} \cdot \quad (6.2)$$

Lemma 6 · 3 [5]: For any $q \in H(div, k)$ we have

$$\sum_{T\in T_h} (-\nabla \cdot q v^0)_T = \sum_{T\in T_h} (\pi_h q, \nabla_{d,r} v)_T, \forall v = \{v^0, v^b\} \in s_h(j,\ell), \quad (6.3)$$

Lemma6 · 4 [19]: For any $u \in H^{1+r}(\Omega)$ with $r > 0$,we have ;

$$\|u - Q^0 u\| \leq Ch^r \|u\|_{1+r} \cdot \quad (6.4)$$

Lemma 6 · 5 [20]: For $u \in H^{1+r}(\Omega)$ with $r > 0$,we have ;

$$\sum_{T\in T_h} (\|u - \pi_h u\| + h^2 \|\nabla u - \pi_h \nabla u\|) \leq Ch^{2(1+r)} \|u\|_{1+r} \cdot \quad (6.5)$$

Lemma 6. 6 [21]: For $u^n \in L^2(0, t, L^{1+r}(\Omega))$ with $r > 0$.We have

$$\|\pi_h(\epsilon\nabla u^n) - \epsilon\nabla_d(Q_h u^n)\| \leq Ch^{1+r} \|u^n\|_{1+r} . \quad (6.6)$$

$$\left\|\nabla u-\nabla_{d,r}(Q_h u^n)\right\| \leq Ch^{1+r}\|u^n\|_{1+r}. \tag{6.7}$$

Theorem 6.1 : Let $u \in H^{1+K}(\Omega)$ with $K>0$ and u_h be the solutions of $(1.1)-(3.1)$ and (4.5) respectively. $Q_h\, u=\{Q^0 u, Q^b u\}$ the L^2 projection of the exact solution $u=u(x,t)$. The error equation of Petrov weak Galerkin finite element method is

$$((u_h - Q_h u)_t\,, v^0)+\left((u_h-Q_h u)_t,\delta\beta\cdot\nabla_{d,r}v\right)+a_{PW}(u_h-Q_h\,u,v)$$
$$=\left(\pi_h(\epsilon\nabla_{d,r}u)-R_h(\nabla u),\nabla_{d,r}v\right)+(cu-c(Q^0 u),v^0)$$
$$+\left(cu-c(Q^0 u),\delta\,\beta\cdot\nabla_{d,r}v\right)+\frac{1}{3}\ \left(u\cdot\nabla_{d,r}u\ -Q^0 u\left(\nabla_{d,r}Qu\right),v^0\right)$$
$$-\frac{1}{3}\left(\pi_h(uu)-Q^0(uu),\nabla_{d,r}v\right)+\frac{1}{3}\left(u\cdot\nabla_{d,r}u-R_h(u)\,,\delta\,\beta\cdot\nabla_{d,r}v\right), \tag{6.8}$$

Proof:

If $v=\{v^0,v^b\}\in\psi_r$ be the testing function .By Lemma 6.3 we obtain ;

$$(u_t\,,v^0)+\left(u_t,\delta\beta\cdot\nabla_{d,r}v\,\right)+\left(\pi_h(\epsilon\nabla_{d,r}u),\nabla_{d,r}v\right)$$
$$(cu,v^0)+\left(cu,\delta\,\beta\cdot\nabla_{d,r}v\,\right)+\frac{1}{3}\ \left(u\cdot\nabla_{d,r}u\,,v^0\right)-\frac{1}{3}\left(\pi_h(uu),\nabla_{d,r}v\right)$$
$$+\frac{1}{3}\left(u\cdot\nabla_{d,r}u\,,\delta\,\beta\cdot\nabla_{d,r}v\right)=(f,v^0\,)+\left(f,\delta\,\beta\cdot\nabla_{d,r}v\right)\cdot \tag{6.9}$$

Adding and subtracting the term

$$a_{PW}(Q_h\,u;Q_h\,u,v)=\left(\epsilon\nabla_{d,r}(Q_h\,u),\nabla_{d,r}v\right)+(c(Q^0 u),v^0)$$
$$+\left(c(Q^0 u),\delta\beta\cdot\nabla_{d,r}v\right)+\frac{1}{3}\ \left(Q^0 u.\left(\nabla_{d,r}Qu\right),v^0\right)\ -\frac{1}{3}\left(Q^0(uu),\nabla_{d,r}v\right)$$
$$+\ \frac{1}{3}\left(u\cdot\nabla_{d,r}(\,Q_h\,u),\delta\,\beta\cdot\nabla_{d,r}v\,\right)\ , \tag{6.10}$$

Using $(Q_h\,u_t,v^0)=(u_t,v^0)$ on the left side of the equation above.

We acquire;

$$(Q_h\,u_t,v^0)+\left(Q_h\,u_t,\delta\beta\cdot\nabla_{d,r}v\,\right)+\left(\pi_h\left(\epsilon\nabla_{d,r}u\right)-\epsilon\nabla_{d,r}(Q_h\,u),\nabla_{d,r}v\right)$$
$$+(cu-c(Q^0 u),v^0)+\left(cu-c(Q^0 u),\delta\,\beta\cdot\nabla_{d,r}v\right)+$$

$$\frac{1}{3}\left(u\cdot\nabla_{d,r}u-Q^0u.\nabla_{d,r}(Q_h\,u),v^0\right)-\frac{1}{3}\left(\pi_h(uu)-Q^0(uu),\nabla_{d,r}v\right)+$$
$$\frac{1}{3}\left(u\cdot\quad\nabla_{d,r}u-Q^0u.\nabla_{d,r}(Q_h\,u)\,,\delta\,\beta\cdot\nabla_{d,r}v\right)=$$
$$(f,v^0)+\left(f,\delta\,\beta\cdot\nabla_{d,r}v\right),\qquad(6.11)$$

and then using $R_h(\nabla u)=\nabla_{d,r}(\,Q_h\,u)$ for $u\in H^1(\Omega)$ and (4.5) we obtain ;

$$\left(u_{h,t}\,,v^0\right)+(u_{h,t}\,,\delta\beta\cdot\nabla_{d,r}v)+a_{PW}(u_h;u_h,v)$$
$$=(Q_h\,u_t,v^0)+\left(Q_hu_t,\delta\beta\cdot\nabla_{d,r}v\right)$$
$$+\left(\pi_h(\epsilon\nabla_{d,r}u)-\epsilon R_h(\nabla u),\nabla_{d,r}v\right)+(cu-c(Q^0u),v^0)$$
$$+\left(cu-c(Q^0u),\delta\,\beta\cdot\nabla_{d,r}v\right)+$$
$$+\frac{1}{3}\left(u\cdot\nabla_{d,r}u-Q^0u.R_h(\nabla u),v^0\right)-\frac{1}{3}\left(\pi_h(uu)-Q^0(uu),\nabla_{d,r}v\right)$$
$$+\frac{1}{3}\left(u\cdot\nabla_{d,r}u-R_h(u)\,,\delta\,\beta\cdot\nabla_{d,r}v\right)+a_{PW}(Q_h\,u;Q_h\,u,v)$$

It can be rewritten as

$$\left(u_{h,t}\,,v^0\right)+(u_{h,t}\,,\delta\beta\cdot\nabla_{d,r}v)+a_{PW}(u_h;u_h-Q_h,v)=$$
$$\left(\pi_h(\epsilon\nabla_{d,r}u)-\epsilon R_h(\nabla u),\nabla_{d,r}v\right)+(cu-c(Q^0u),v^0)$$
$$+\left(cu-c(Q^0u),\delta\,\beta\cdot\nabla_{d,r}v\right)+\frac{1}{3}\left(u\cdot\nabla_{d,r}u-Q^0u.R_h(\nabla u),v^0\right)-$$
$$\frac{1}{3}\left(\pi_h(uu)-Q^0(uu),\nabla_{d,r}v\right)+\frac{1}{3}\left(u\cdot\nabla_{d,r}u-R_h(u)\,,\delta\,\beta\cdot\nabla_{d,r}v\right),\qquad(6.12)$$

The proof is complete. ■

Theorem 6.2:

Let u^{n} and u_h^n be the solution of (4.5) and (4.6) respectively, then there exists a constant C independent of h such that;

$$\|e\|^2+\int_0^t\|e_t\|^2\leq Ch^{2k}\int_0^t\|u\|_{1+k}^2,\qquad(6.13)$$

Proof:

Let $e=\{u_h-Q_hu\}$, $\{e^0,e^b\}=\{u^0-Q^0u,\ u^b-\ Q^bu\}$, and put $v^0=e$ in equation (6.11) we have ;

$$(e_t\,,e)+\left(e_t,\delta\beta\cdot\nabla_{d,r}e\right)+\ a_{PW}(e;e,e)$$
$$=\left(\pi_h(\epsilon\nabla_{d,r}u)-\epsilon R_h(\nabla u),\nabla_{d,r}e\right)+(cu-c(Q^0u),e)$$

$$+\left(cu - c(Q^0 u), \delta\,\beta \cdot \nabla_{d,r}\, e\right) + \frac{1}{3}\ \left(u \cdot \nabla_{d,r} u\ - Q^0 u. R_h(\nabla u), e\right) -$$

$$\frac{1}{3}\left(\pi_h(uu) - Q^0(uu), \nabla_{d,r} e\right) + \frac{1}{3}\left(u \cdot \nabla_{d,r} u - R_h(u)\ , \delta\,\beta \cdot \nabla_{d,r} e\right), \qquad (6.14)$$

Cauchy-Schwartz inequality, Lemma 6.3, and Young's inequality of the bilinear form lead to the following results;

$$\frac{1}{2}\frac{d}{dt}\|e\|^2 + \frac{1}{2}\|e\|_{Pw}^2\ + \|e_t\|^2 + \frac{1}{4}\delta\left\|\beta \cdot \nabla_{d,r}\ \theta e\right\|^2 = \sum_{i=1}^{6} I^i\ \ , \qquad (6.15)$$

Where

$$I^1 = \left(\pi_h(\epsilon \nabla_{d,r} u) - \epsilon R_h(\nabla u), \nabla_{d,r} e\right)$$

$$I^2 = (cu - c(Q^0 u), e)$$

$$I^3 = \left(cu - c(Q^0 u), \delta\,\beta \cdot \nabla_{d,r}\ e\right)$$

$$I^4 = \frac{1}{3}\ \left(u \cdot \nabla_{d,r} u\ - Q^0 u. R_h(\nabla u), e\right)$$

$$I^5 = -\frac{1}{3}\left(\pi_h(uu) - Q^0(uu), \nabla_{d,r} e\right)$$

$$I^6 = \frac{1}{3}\left(u \cdot \nabla_{d,r} u - R_h(u)\ , \delta\,\beta \cdot \nabla_{d,r} e\right).$$

Application the Cauchy-Schwartz inequality and Young's inequality, we may estimate I^1 and arrive at;

$$|I^1| \le \frac{1}{2\xi}\ \|\pi_h(\epsilon \nabla u) - \epsilon R_h(\nabla u)\|^2\ + \frac{\xi}{2}\ \left\|\nabla_{d,r} e\right\|^2,$$

Through Lemma 6.1, we get;

$$|I^1|\ \le C h^{2k}\ \|u\|_{1+k}^2 + \frac{\xi}{2}\ \left\|\nabla_{d,r} e\right\|^2\ \ . \qquad (6.16)$$

To estimate I^2, we get the following results using the Cauchy-Schwartz inequality, using Young's- inequality, and Lemma 6.2, we get;

$$|I^2|\ \le \frac{1}{2\xi}\ \ \|cu - c(Q^0 u)\|^2 + \frac{\xi}{2}\ \|e\|^2$$

$$|I^2|\ \le C h^{2k}\ \|u\|_{1+k}^2 + \frac{\xi}{2}\ \ \|e\|^2 \cdot \qquad (6.17)$$

To estimate I^3 , we get the following results using the Cauchy-Schwartz inequality, Young's inequality, and Lemma 6.1, we get;

$$|I^3|\ \le \frac{1}{2\xi}\ \ \|cu - c(Q^0 u)\|^2 + \frac{\xi}{2}\ \ \delta\left\|\beta \cdot \nabla_{d,r} e\right\|^2$$

$$|I^3| \le Ch^{2k}\|u\|_{1+k}^2 + \frac{\xi}{2} \quad \delta\|\beta\cdot\nabla_{d,r}e\|^2 \quad \cdot \tag{6.18}$$

Then using $R_h(\nabla u) = \nabla_{d,r}(\ Q_h\ u)$, we can obtain;

$$I^4 + I^5 = \frac{1}{3}\ \left(u\cdot\nabla_{d,r}u - Q^0u.R_h(\nabla u), e\right) - \frac{1}{3}\left(\pi_h(uu) - Q^0(uu), \nabla_{d,r}e\right)$$

we rewritten

$$I^4 + I^5 = \frac{1}{3}\ \left(u\cdot\nabla_{d,r}u - Q^0u.R_h(\nabla u), e\right) - (-(\frac{1}{3}\left(Q^0(uu) - \pi_h(uu), \nabla_{d,r}e\right) =$$

$$\frac{1}{3}(\ u\cdot\nabla_{d,r}u, e)\ - \frac{1}{3}(Q^0u.R_h(\nabla u), e) + \frac{1}{3}\left(Q^0(uu), \nabla_{d,r}e\right) - \frac{1}{3}\left(\ \pi_h(uu), \nabla_{d,r}e\right), \tag{6.19}$$

and, then using $R_h(\nabla u) = \nabla_{d,r}(\ Q_h\ u)$, we obtain;

$$\frac{1}{3}(\ (u(\nabla_{d,r}u - \nabla_{d,r}(\ Q_h\ u), e) + (\frac{1}{3}\left((uu) - \ \pi_h(uu), \nabla_{d,r}e\right) = R_1 + R_2 \tag{6.20}$$

To estimate R_1 by Young's-inequality and Lemma 6.6, we obtain;

$$|R_1|\ \le \frac{1}{3}(\frac{5}{12}\|u\|^2\ \|\nabla u - \nabla_{d,r}(Q_hu^n)\| + \frac{3}{5}\ \ \|e\|^2\cdot \tag{6.21}$$

$$\le Ch^{2K}\|u^n\|_{1+K}\ +\ \frac{1}{5}\ \ \|e\|^2\cdot \tag{6.22}$$

From [8] , we rewritten R_2 as follows:

$$|R_2|\ \le \frac{1}{3}\frac{d}{dt}\left(uu - \ \pi_h(uu)\right), \nabla_{d,r}e - \frac{1}{3}\ \left(\frac{d}{dt}uu - \ \pi_h(uu), \nabla_{d,r}e\right)\cdot \tag{6.23}$$

To estimate R_2 by Cauchy – Schwartz and Lemma 6.5, we obtain;

$$|R_2|\ \le \frac{1}{3}\frac{d}{dt}\left(uu - \ \pi_h(uu)\right), \nabla_{d,r}e + \frac{1}{6}\ \left\|\frac{d}{dt}uu - \ \pi_h(uu)\right\|^2 + \frac{1}{6}\|\nabla_{d,r}e\|^2, \tag{6.24}$$

$$\le \frac{1}{3}\frac{d}{dt}\left(uu - \ \pi_h(uu), \nabla_{d,r}e\right) + Ch^{2K}\|u^n\|_{1+K} + \frac{1}{6}\|\nabla_{d,r}e\|^2, \tag{6.25}$$

and, by Cauchy – Schwartz and Lemma 6.5, we obtain;

$$\frac{1}{3}\frac{d}{dt}\left(uu - \ \pi_h(uu), \nabla_{d,r}e\right)\ \le Ch^{2K}\|u^n\|_{1+K} + \frac{1}{6}\|\nabla_{d,r}e\|^2, \tag{6.26}$$

To estimate I^6 using Young's-inequality and Lemma 6.6, we get ;

$$I^6 \le \frac{1}{3}(\frac{5}{12}\|u\|^2\ \|\nabla u - \nabla_{d,r}(Q_hu^n)\| + \frac{3}{5}\ \delta\|\beta\cdot\nabla_{d,r}e\|^2 \quad \cdot$$

$$\le Ch^{2K}\|u^n\|_{1+K}\ +\ \frac{1}{5}\ \delta\|\beta\cdot\nabla_{d,r}e\|^2 \tag{6.27}$$

Substituting (6.16), (6.17), (6.18), (6.22) ,(6.24),(6.25),(6.26) and(6.27) in to (6.5) we get;

$$\frac{1}{2}\frac{d}{dt}\|e\|^2+\frac{1}{2}\|e\|_{Pw}^2+\|e_t\|^2+\frac{1}{4}\delta\left\|\beta\cdot\nabla_{d,r}\,\theta e\right\|^2$$

$$\leq Ch^{2k}\|u\|_{1+k}^2+\frac{\xi}{2}\left\|\nabla_{d,r}e\right\|^2+\frac{\xi}{2}\|e\|^2$$

$$+\frac{\mu}{2}\ \delta\left\|\beta\cdot\nabla_{d,r}e\right\|^2+\frac{1}{5}\ \|e\|^2+\frac{1}{3}\frac{d}{dt}\left(uu-\pi_h(uu)\right),\nabla_{d,r}e)$$

$$+\frac{1}{6}\left\|\nabla_{d,r}e\right\|^2+\frac{1}{5}\ \delta\left\|\beta\cdot\nabla_{d,r}e\right\|^2, \qquad (6.28)$$

It follows that:

$$\frac{1}{2}\frac{d}{dt}\|e\|^2+\frac{1}{2}\|e\|_{Pw}^2+\|e_t\|^2\leq Ch^{2k}\|u\|_{1+k}^2$$

$$+\alpha\left\|\nabla_{d,r}e\right\|^2+\rho\|e\|^2+\eta\ \delta\left\|\beta\cdot\nabla_{d,r}e\right\|^2, \qquad (6.29)$$

Where $\alpha=\frac{\xi}{2}+\frac{1}{6}+\frac{1}{6},\rho=\frac{\xi}{2}+\frac{1}{5}$, $\eta=\frac{\xi}{2}+\frac{1}{5}-\frac{1}{4},$ since

$$\alpha\left\|\nabla_{d,r}e\right\|^2+\rho\|e\|^2+\eta\ |\delta|\left\|\beta\cdot\nabla_{d,r}e\right\|^2\leq\frac{1}{2}\|e\|_{Pw}^2,$$

we have

$$\frac{1}{2}\frac{d}{dt}\|e\|^2+\left(\frac{1}{2}-\frac{1}{2}\right)\|e\|_{Pw}^2+\|e_t\|^2$$

$$\leq Ch^{2k}\|u\|_{1+k}^2\quad, \qquad (6.30)$$

Thus, using the integral on both sides from (0 to t), we have;

$$\|e\|^2+\int_0^t\|e_t\|^2\leq Ch^{2k}\int_0^t\|u\|_{1+k}^2, \qquad (6.31)$$

7.1. Numerical Results:

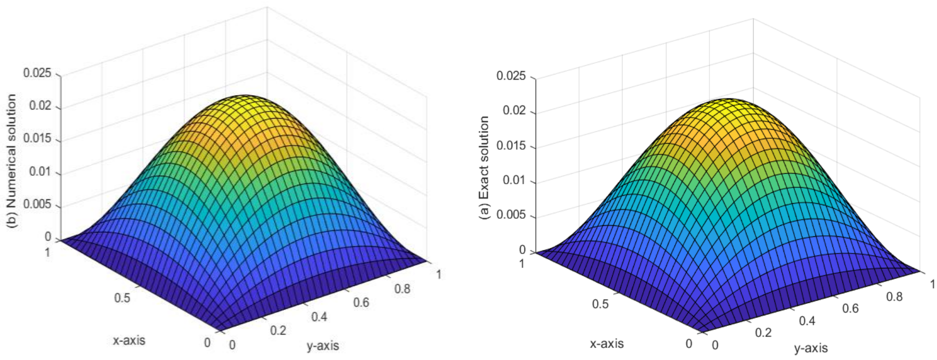
We analyze the nonlinear convection-diffusion problem (1.1)- (1.3) in $\Omega=[0,1]\times[0,1]$, where the time period is (0,T)=(0,1) and the exact solution is set to define the various diffusion coefficients $\epsilon=0\cdot01,0\cdot001$ (c=1, f, and g).[8]

$$u=x(1-x)y(1-y)\exp(-t)\cdot$$

It employs the uniform triangular partition, where $h={}^{1}/_{N}$ $(N=2,4,8,16,32)$ denotes the size of the locative mesh. For the time discretization, we also employ the uniform partition., where T stands for the time step. After fixing a small enough time step, T=0.001, the error of $u-u_h$ for L^2 and H^1 with reference to the mesh size can be found in tables (2) and (4), It supports the theoretical analysis as well. Figure 1 displays the numerical response from PWG and precise solution for the mesh size $h = \frac{1}{32}$. The PWG solution and exact solution are shown in Figure 2.

Table (1): $H^1 - error$ and $L^2 - error$ for PWGFEM with $\mathcal{T} = 0 \cdot 001$, $\epsilon = 0 \cdot 1$

h	$H^1 - error$	$H^1 - order$	$L^2 - error$	$L^2 - order$
$\frac{1}{2}$	0.345116	0	0.098968	0
$\frac{1}{4}$	0.1822312	0.92131	0.032581	1.6029
$\frac{1}{8}$	0.0924353	0.97925	0.008235	1.9841
$\frac{1}{16}$	0.047814	0.95101	0.002118	1.9590
$\frac{1}{32}$	0.091645	0.93862	0.0009011	2.0889

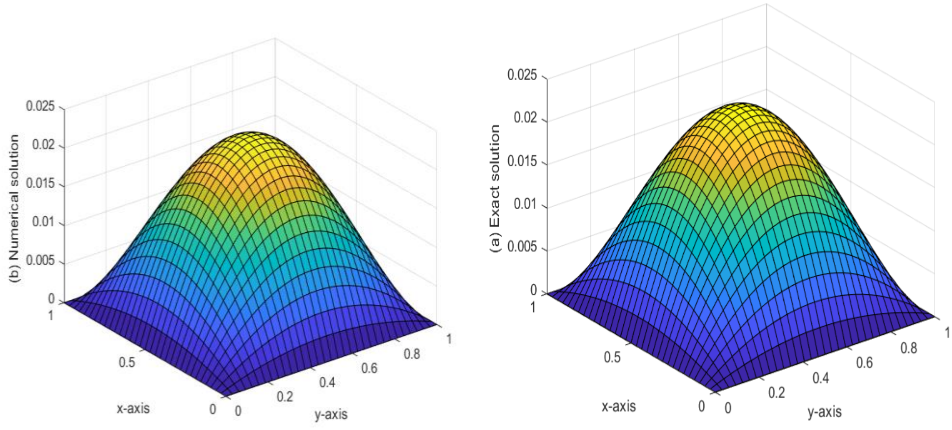


(b) (a)

Fig 1. (a) WG's exact solution with $\epsilon = 0 \cdot 1$. (b) the PWG's numerical response with $\epsilon = 0 \cdot 1$

Table (2): $H^1 - error$ and $L^2 - error$ for PWGFEM with $\mathcal{T} = 0 \cdot 001$, $\epsilon = 0 \cdot 01$.

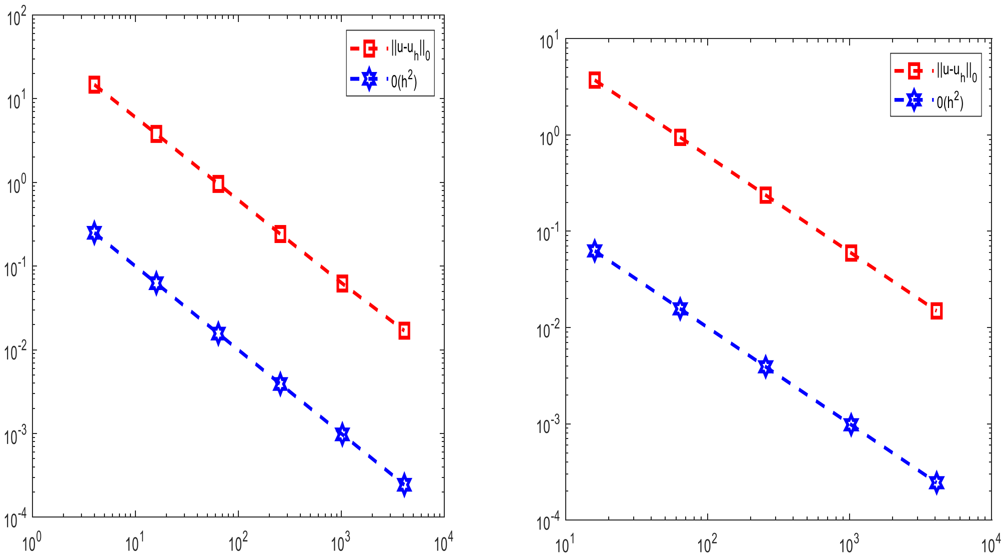
h	$H^1 - error$	$H^1 - order$	$L^2 - error$	$L^2 - order$
$\frac{1}{2}$	0.1836445	0	0.096961	0
$\frac{1}{4}$	0.096991	0.9209	0.031095	1.64072
$\frac{1}{8}$	0.051126	0.9237	0.008199	1.92316
$\frac{1}{16}$	0.026689	0.9378	0.002115	1.9547
$\frac{1}{32}$	0.013887	0.94251	0.0005419	1.9669



(b) (a)

Fig 2. (a) WG's exact solution with $\epsilon = 0 \cdot 01$. (b)the PWG's numerical response with$\epsilon = 0.001$

$L^2 - error$ $L^2 - error$



(b) (a)

Figure 3. (a) With $\epsilon = 0 \cdot 1$, PWG Convergence Rate Convection – Diffusion.

(b) With $\epsilon = 0 \cdot 01$, PWG Convergence Rate Convection – Diffusion.

8: Discussion and Conclusion:

The semi-discrete PWG finite element method is considered in this study for the nonlinear convection-diffusion problem. As a result of our findings, we characterize that the PWGFE approach is noticeably more precise than the WG. Additionally, we demonstrate that PWG mothed according to the numerical tests; see figure (3). demonstrated the consistency of the exact solutions and numerical findings for the unsteady-state PWGFE mothed.

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