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Heat Transfer in Solids. Dual Phase Lag, Thermal Stationary Waves and the Development of Thermal Strings Model: Masonry Wall Case Study

S Veranoudis[1], S Zoras[1*], A Dimoudi[1]

1. Department of Environmental Engineering, Democritus University of Thrace, Xanthi, Greece

*Corresponding author S Veranoudis (sveranou@env.duth.gr)

Abstract

The purpose of this study is the development of an analytical solution based computational tool for heat transfer in solids. The proposed method extends the Dual Phase Lag theory by the introduction of the solid body that is considered as a pulsating thermal string whose oscillations correspond to temperature fluctuations. These oscillations give stationary thermal waves in every axis. The combination of the three dimensional 3D Thermal Strings provided the ability to describe the temperature changes of solid bodies, the calculation of the relaxation time required towards equilibrium, and the thermal load required to keep a non-equilibrium state. The model is based on the analytical solution of the heat transfer equation in three dimensions, so it can deal for the whole solid body e.g. a wall element. It is proved that these oscillations are critical dumped ones and the model uses mathematical formulations to calculate the time required for an area to reach a "target temperature" which describes the final temperature. The solution depends on the initial and boundary conditions of the solid. The validation and reliability of the Thermal Strings model were obtained by comparing the results of the proposed model against the classic heat transfer which found to be identical.

Keywords: Dual Phase Lag, Heat transfer, Thermal Strings, Non-Fourier heat conduction

1. Introduction

An important problem related to the properties of solid bodies is the heat transfer through them and their response to temperature variations that occur around them. If a solid is in thermal equilibrium, means that it has reached a constant temperature. This state can be permanent and stable or unstable and dynamic. In the first case, when the body spontaneously ends up in this state, no further transfer of energy occurs, while in the second, a continuous flow of heat is required from one region of the solid to another.

If the temperature is changing in one area on a free surface of the solid, then the equilibrium is disrupted. The solid will try to reach a new equilibrium condition. The temperature distribution will change and the body will reach a new condition after some time. The study of the phenomenon is complicated due to the requirement of the knowledge of the thermal "history" of the body, and detailed calculations for each part within the volume of the solid.

Various models and methods have been proposed [1] to solve the problem. Some of them refer to one or two dimensions [2-5] solving the problem over surfaces and not over the whole volume. Most of them use approximate methods (finite difference, finite elements method) [6-8], while there are others that perform analytical solutions of the models [9-10], or solutions through analogies such as electrical or mechanical models [11].

The difficulty of solving heat transfer equations relies on the following:

i) the theoretical model. Sometimes the model seems to work properly only within certain limits.

ii) the initial and boundary conditions of the problem (steady state, variable temperatures, external effects, geometry etc.).

iii) the mathematics (analytical, numerical, or computational methods and assumptions, sometimes, may lead to discrepancies and inaccurate results).

There are three models for heat transfer in solids. A) The Fourier heat diffusion model. B) The Cattaneo-Vernotte thermal wave model and C) The Dual Phase Lag model.

Fourier model

The heat behaves like a fluid and is transferred from the cold part to the hot part by a diffusion mechanism. The heat is considered as a fluid that expands to the cold parts like a small amount of red liquid spreading in a glass of water or a perfume in a room.

Cattaneo-Vernotte's model

The heat is not transferred by diffusion but in the same way that oscillations do. Oscillations are transferred with waves. In this kind of mechanism the heat is also transferred between the hot and the cold part of a solid.

Dual Phase Lag-DPL model

It is an extension of Cattaneo-Vernotte's model that describes in a better way the mechanism of heat transfer as a wave.

The difference between the three models is the time. In Fourier's model there is no time delay. If three points on a line in the solid have different temperatures, then the heat that is transferred from A to B and from B to C is proportional to the temperature difference and the response speed of the solid is infinite. The greater the temperature difference (u_C-u_A), the more heat is transferred. The heat transfer from A to B occurs at the same time with the heat transfer from B to C, without delay. B and C points start to receive heat at the same time and therefore begin to heat up at the same time.

In Cattaneo-Vernotte's model there is time delay at the start of heating up the C point. That is why there is a characteristic time which declares that the heat delays to flow forward. This happens because the point B must increase its temperature and has to absorb enough heat, before it allows the heat to move to point C. The heat is transferred like a vibration on a string and propagates like a wave. The oscillation here is not a variation of the position of the particle, not a displacement, but a variation of its temperature from its initial value. Thus, we have a temperature disturbance [12].

In DPL model there are two characteristic times, describing two kinds of time delays. The one deals with the time delay of heat transfer and is the same with the characteristic time in Cattaneo-Vernotte's model. The other is the delay time of temperature change of the material. This declares that the material needs some extra time to "realize" the arrival of heat from B to C. This causes a delayed temperature increment [13-14], that needs some extra time to start until it stops changing, even if the whole amount of heat has already arrived. That becomes the reason of some extra amount of heat absorbed from the area. This extra heat creates a regression or reflux of the wave.

The DPL model has been proved right for small time periods, for microscale effects and fast-transient effects [15-16]. There is an extended discussion about the two delay-times. Whenever the one time is greater than the other, the temperature difference is the cause and the heat flux is the effect or vice-versa [17]. The two phenomena work simultaneously and competitively.

The heat equations that describe the three models are:

$$q = -k_\sigma \cdot \nabla u, \quad \text{(Fourier)} \tag{1}$$

$$q + \tau \cdot \frac{\partial q}{\partial t} = -k_\sigma \cdot \nabla u, \quad \text{(Cattaneo-Vernotte)} \tag{2}$$

$$q + \tau_o \frac{\partial q}{\partial t} = -k_\sigma (\nabla u + \tau_T \frac{\partial}{\partial t}(\nabla u)), \quad \text{(DPL)} \tag{3}$$

For the study of heat transfer, a basic equation is the one of thermal energy storage derived from the principle of conservation of energy:

$$\nabla q = -\rho c_{sp} \frac{\partial u}{\partial t}, \quad (4)$$

This equation states that the heat flux varies locally due to absorption of heat by the material. This absorption creates a temporal change of temperature in the area. Definition of the symbols of the above equations follows:

q the heat per unit of time and per unit of area (W/m^2).

u temperature (K).

t time (sec).

ρ density of the material (kg/m^3).

c_{sp} specific heat capacity ($J \cdot K^{-1} \cdot kg^{-1}$).

k_σ the coefficient of thermal conductivity of the solid body (W/m/K).

τ, τ_o, τ_T characteristic delay-times

All the methods of solving the problem of heat transfer need to handle effectively the boundary and the initial conditions. The τ and τ_o times are the same in Equations 2 and 3 and describe the delay of heat propagation. The τ_T characteristic time describes the delay of temperature due to material's thermal inertia.

It is assumed a perfect insulated solid cuboid body with its one surface having high temperature and the opposite one having low temperature. The half of the body is a "hot tank" and the other half is a "cold tank". If the length of the body is L, then at x=L/2 will have heat flux arriving from the hot part, which is $q_H = -k_\sigma \cdot \frac{\Delta u}{L/2} = -k_\sigma \cdot \frac{u_h - u}{L/2}$ and heat flux leaving to the cold part $q_C = -k_\sigma \cdot \frac{u - u_c}{L/2}$ which have to be equal.

So, $q_H = q_C \Rightarrow -k_\sigma \cdot \frac{u_h - u}{L/2} = -k_\sigma \cdot \frac{u - u_c}{L/2} \Rightarrow u = \frac{u_h + u_c}{2}$. The temperature at the middle point will be equal to the average of the temperatures of the hot and the cold part due to symmetry.

The final temperature of the body would be $u\left(\frac{L}{2}\right) = \frac{u_h + u_c}{2} = u_{final} = u_{average}$. Then, a function can be defined which will be equal to the difference of the temperature of each point from its final temperature in time. Along x axis function would become $f(t,x) = u(t,x) - u_{final}(x)$. If all the points would reach at the same final temperature, then $u_{final}(x) = u_f$ for 0≤x≤L thus, $f(t,x) = u(t,x) - u_f$. For 3D solids it is generalized to $f(t,x,y,z) = u(t,x,y,z) - u_f$. That, according to the classical theory, the f function is linear.

If the boundary conditions are Dirichlet's type, then the boundaries will have steady temperature. If they are Neumann's type will have steady heat flux. In case of perfect insulated body the flux will be zero. So, $q(0) = q(L) = \frac{\partial u(0)}{\partial x} = \frac{\partial u(L)}{\partial x} = 0$. A solution that satisfies these boundary conditions is the known distribution with the help of the error function $f = \Theta_o[1 - erf\left(\frac{x - \frac{L}{2}}{2\sqrt{at}}\right)$. This equation describes the distribution in case of two bodies (hot and cold) being in contact at x=L/2.

2. The temperature distribution

2.1. The energy transfer mechanism

The classical theory says that the cold side takes zero time to "feel" any thermal disturbance that occurs at the hot side. The heat begins to flow simultaneously to all the areas with no delay. The speed of the "feeling" of the signal of the thermal anomaly is infinite. DPL theory orders that the body has thermal inertia so that temperature starts changing at the heat departure area and then (after a time delay) the temperature variations start to occur at the heat arrival area. The DPL model solves the problem of the infinite heat transfer velocity, but this model is valid for very small time periods (10^{-10}-10^{-14}sec).

In the body there are isothermal surfaces which at the beginning are dense due to the high temperature grade. These isothermals are becoming less dense as time passes, because of the temperature difference decreasing. The density of the isotherms is decreasing with the same speed due to symmetry except the central area that remains unchanged under steady isotherm through the end of the process. At the DPL model a displacement of the isotherms which is not symmetrical takes place. The central isotherm seems to be displaced so the stability of the central area is lost. At the end, the symmetry is re-established and the whole area reaches the same temperature.

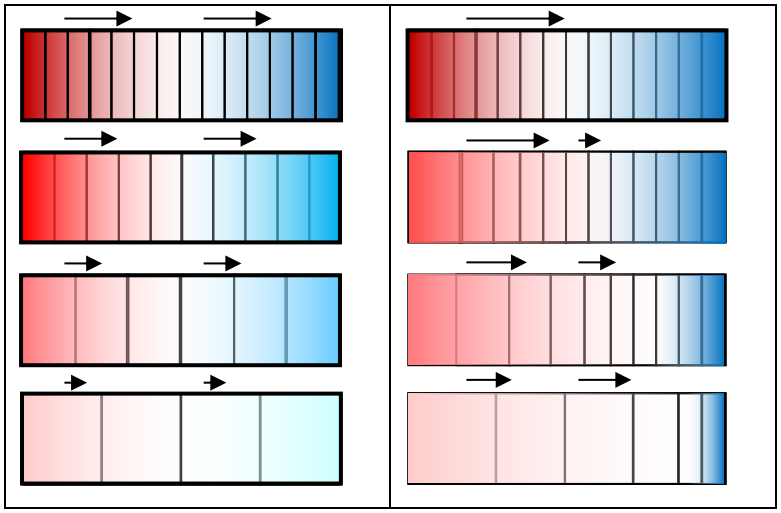

Figure 1: Isotherms inside a solid body during reaching equilibrium. Classical model (left) vs DPL model (right)

In Figure 1 (heat map of the solid body) the arrows represent the heat flux. At the classical model the heat flux is the same everywhere and decreases in the same way by time. According to the DPL model the flux is high at the departure area and later is been increasing at the cold areas. The delay is because of the thermal inertia so that no area remains steady.

Towards, the solid body heat makes a step forward and has to do two jobs. The first job is to increase the temperature to the point it just arrived at. The other job is to continue its way forward so to heat the rest of the solid. But what is the fraction of the absorbed energy at the area and what is the fraction of heat to continue forward? Is it possible the whole amount of energy to be absorbed in a small area so to just increase the temperature without affecting the neighbouring points? And how much would be the increment of the temperature until the material would allow the heat to keep moving to the next cold area? Is there a limit that beyond this a “heat overflow” occurs? What is the heat capacity of each area and what would have happened if it was kept providing heat from an outer source to the edge of the solid? Is it possible to restrict the heat staying at this narrow limited area and keep increasing continuously the temperature but restricting further heat transfer beyond this point? Is it possible to create heat pulse affecting only a small area? How small this could be? Perhaps, the area covered of some molecules? Is the Gaussian function quite enough to describe the amplitude and the range of a heat pulse? Are the molecules independent or part of a long chain of molecules?

If the temperature delay time is greater than the flux delay time, then the temperature elevation is delayed, and this will be the reason for an additional heating which would, ultimately, cause temperature increment above the “proper” final value.

If the temperature delay time is shorter than the flux delay time, then, in the above example with the cuboid body, the hot part of the body will decrease its temperature, thus would lose internal energy, but this lost energy does not seem to flow anywhere, because no flux is being counted for a while.

In conclusion, the two competitive phenomena (heat storage vs heat flux) determine the velocity of heat transfer by two physical quantities, the specific heat capacity and the coefficient of thermal conductivity which describe the behavior of the material. Hence, the delay times (τ_o, τ_T) control the two mechanisms and could provide a resulted “heat transfer velocity” value.

2.2. Diffusion versus Wave

In the phenomenon of heat conduction with Dual Phase Lag - DPL (Tzou 1995) the Fourier equation transforms into:

$$q(r, t_q) = -k_\sigma \nabla u(r, t_T)$$

or

$$q(r, t + \tau_o) = -k_\sigma \nabla u(r, t + \tau_T)$$

The heat transfer Equation 3 in three dimensions of Dual Phase Lag combined with Equation 4 gives:

$$\frac{\partial^2 u}{\partial t^2} + \frac{1}{\tau_o}\frac{\partial u}{\partial t} = c^2 \nabla^2 u + c^2 \tau_T \frac{\partial}{\partial t}(\nabla^2 u), \qquad (5)$$

In Equation 5 it was set $c^2 = \frac{k_\sigma}{\rho c_{sp} \tau_o}$. The quantity $c = \sqrt{\frac{k_\sigma}{\rho c_{sp} \tau_o}}$ has units of speed and is equivalent to the velocity of heat propagation through the material.

It is noted that if $\tau_\tau = \tau_o$ then the Dual Phase Lag equation has the same format with the Fourier's equation $q(r, t + \tau_o) = -k_\sigma \nabla u(r, t + \tau_T) \Rightarrow q(r, t') = -k_\sigma \nabla u(r, t')$ but they describe two discrete and different ways of heat transfer.

The formula of the heat velocity $c = \sqrt{\frac{k_\sigma}{\rho c_{sp} \tau_o}}$ gives a comprehensive description of the two "operations" of the transferred heat: the storage (ρc_{sp}) and the conduction (k_σ). It is assumed that the quantity τ_o is a ratio factor between the two mechanisms that describes the characteristic time for the material. There is also a special number (Fourier Number) which shows whether conduction prevails over storage and for its calculation it is used a diffusion time scale which is equal to $t_d = \frac{\rho c_{sp} L^2}{k_\sigma}$ and it is multiple of the characteristic time τ_o.

The classical model assumes that the temperature gradient has to be linear and steady. If the distribution wasn't linear then the temperature had to change in order to obtain normal range. This can be met with the heat flux with the middle points already having the "proper" temperature. That's why they do not have to change their temperature and simply they pass the energy onto the next area. So, if it is accepted the equation of the classical theory then, inevitably, it is accepted the existence of steady points. Ultimately, this is what truly happens because all macroscopically know that the middle points do not change their temperature as they don't have to.

If it is accepted the DPL theory then there is a delay of "feeling" the temperature disturbance. Because of the inertia, the material takes time to understand that the heat has arrived and already has gone forward. The middle point does not realize that has reach at its right condition and its temperature increases beyond equilibrium value. So, due to DPL theory, the middle points would not be points of fixed temperature and there would not be symmetrical temperature distribution during the whole period of the phenomenon. This is actually the difference between the two models.

Another difference of the DPL theory is that the behavior of the material changes with time relies on heat transferred by the wave mechanism for a short time period. After this point it is not known yet how the solid allows the heat to be transferred only by diffusion and how can be possible for the material to change its properties depending on the duration of the phenomenon?

Effectively, it is here claimed the following necessary conditions to compromise: a) finite velocity of thermal signals b) points with steady temperature c) steady behavior of the material (not depending on the duration of the conduction). It is the actual mechanism between the two theories explaining quite well the phenomena and that the one is a special case of the other theory. An efficient way to solve the specific issue is to consider that the two delay times are equal which leads to similar equation representation of the two theories. The wave induces changes in temperature as fast as the Fourier theory and leaves the intermediate points at fixed temperatures. Then, stationary waves are generated due to wave regression and leave fixed points in the medium. Due to inertia (time τ_T) reverse heat waves are generated and the material reaches a temperature higher than its neighboring points. this is because each point is delayed not only in increasing its temperature, but also

delayed in ceasing to increase beyond the point it should. This causes an amount of heat going backwards while another flows forwards. Therefore, at each point we have two heat waves that move in opposite directions and are identical. Their interference generates a stationary wave with equal delay times which is an ultimate analogy of the two equations describing Fourier's and DPL models. Therefore, it is claimed here a new approach that solves the issue from the end to the beginning.

3. Thermal Strings Model

3.1. The fundamental equation

It is defined a function f(t,x,y,z) that would not express the temperature of the points of the material medium, but the temperature difference of the points from their final value. So $f = u - u_f$ and therefore $u = f + u_f$.

Based on the latter it is obtained:

$$\frac{\partial u}{\partial t} = \dot{u}, \qquad \frac{\partial^2 u}{\partial t^2} = \ddot{u}, \qquad \frac{\partial f}{\partial t} = \dot{f}, \qquad \frac{\partial^2 f}{\partial t^2} = \ddot{f}$$

$$\dot{f} = \dot{u}, \qquad \ddot{f} = \ddot{u}, \qquad \nabla^2 u = \nabla^2(f + u_f) = \nabla^2 f, \quad \frac{\partial}{\partial t}(\nabla^2 u) = \frac{\partial}{\partial t}(\nabla^2 f)$$

This means that the behavior of the function f is similar to the behavior of the function u. Therefore, the differential Equation 5 is transformed into:

$$\ddot{f} + \frac{1}{\tau_o}\dot{f} = c^2\nabla^2 f + c^2\tau_T \frac{\partial}{\partial t}(\nabla^2 f), \tag{6}$$

Then it was divided the cuboid into planes and then shred into lines, in order to work with heat propagation only along the x-axis (Figure 2). Equation 6 would have a solution of $f = \Theta(t) \cdot X(x)$. Therefore

$$\ddot{\Theta} \cdot X + \frac{1}{\tau_o}\dot{\Theta} \cdot X = c^2\Theta X'' + c^2\tau_T\dot{\Theta}X'' \Rightarrow \frac{X''}{X} = \frac{\ddot{\Theta} + \frac{1}{\tau_o}\dot{\Theta}}{c^2\Theta + c^2\tau_T\dot{\Theta}} = -\lambda$$

where

$$X'' + \lambda X = 0, \tag{7}$$

$$\ddot{\Theta} + \left[\frac{1}{\tau_o} + \lambda c^2\tau_T\right]\dot{\Theta} + \lambda c^2\Theta = 0 \tag{8}$$

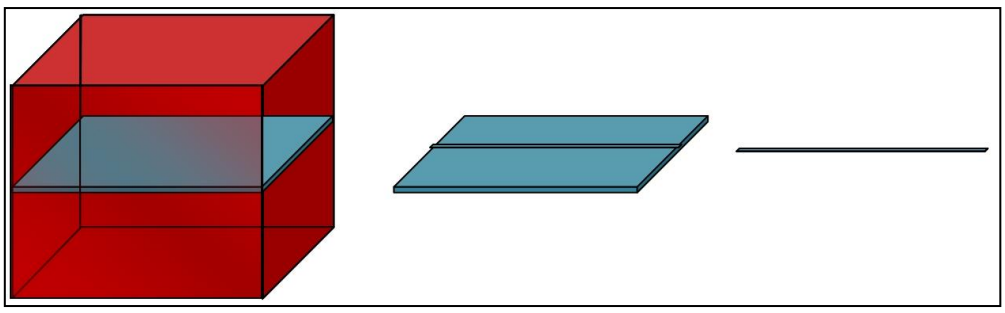

Figure 2: Shredding a solid to surfaces and strings

The form of the time solution is not implied from the time equation, but from the spatial one and the solution is not a consequence of the study of the transferred energy, but of the necessity to explain its result to the temperature distribution.

3.2. Spatial solutions

The spatial solutions would have the form of $X_{(x)} = A \cdot cos\,(bx + \varphi_o)$ and would be defined using the eigenvalues of b and λ depending on the boundary conditions of the problem. There are three cases of boundary conditions:

a) The opposite surfaces are temperature free (unsteady). This means that finally they will reach equilibrium at a target or final temperature given as $u_f = \frac{u_h+u_C}{2}$.

b) The one surface is in steady state condition while the other is free. If the hot surface has a constant temperature then $u_f = u_h$. If the cold surface has constant temperature, then $u_f = u_C$.

c) Both surfaces keep their initial temperatures and they would not reach a common temperature u_f.

In all cases of those boundary conditions, there are points of constant temperature that along a specific axis, the fixed point would always be "one" with the other points on this string would experiencing the temperature fluctuations. The initial form of the fluctuations will be in the form of a stationary wave described by a node point.

The spatial solutions would be given as $X_{(x)} = A \cdot cos\ (bx + \varphi_o)$ with f given as :

$$f = \Theta(t) \cdot X(x) = \Theta(t) \cdot A \cdot cos\ (bx + \varphi_o)$$

The b and φ_o depend on the boundary conditions. If there were more points with constant temperature along the string, then the function f would have a more complex form. For the case of free boundaries it would be

$$f(t,x) = \Theta(t) \cdot A \cdot \cos\left(\frac{2n\pi+\pi}{L}x + \varphi_o\right) \quad \text{with n=0, 1, 2…}$$

and in current case (only one node) n=0.

3.3. Time solutions

The time solutions arise from Equation 8 which is a damped oscillation equation. Damping oscillations are three kinds: underdamped, overdamped, and critical damped.

Overdamped oscillations are rejected because they are abrupt and the system does not have time to return to its equilibrium state. So, it would never reach its final temperature which is generally impossible.

If it is set $\tau_\tau=\tau_o$ at Equation 8, then, it is impossible for the oscillation to be underdamped.

Thus, the only solution is the critical damped which is not a coincidence, but a necessary condition, as long as it is accepted that the DPL model is reliableand the Fourier model is a subcase of the DPL assuming that there are points of constant temperature on a string, and that thermal stationary waves are generated on the string. After all, critical damping is the shortest path for a damping oscillation to reach its final value.

Equation 8 becomes $\ddot{\Theta} + \left[\frac{1}{\tau_o} + \lambda c^2 \tau_o\right]\dot{\Theta} + \lambda c^2 \Theta = 0$ with the following solution:

$$\Theta(t) = \Theta_o(1 + \omega_o t)e^{-\omega_o t}$$

In fact, it turns out that $\omega_o = \frac{1}{\tau_o}$ and :

$$\Theta(t) = \Theta_o(1 + \frac{t}{\tau_o})e^{-\frac{t}{\tau_o}}$$

3.4. General solution of Thermal strings model

If it is combined the spatial with the time solution of the problem, an integral form of the final equation is generated. In each case of boundary conditions, the solutions on the x-axis are:

$$\mathbf{f_x(t,x) = u(t,x) - u_{fx} = \Theta_{ox}\left(1 + \frac{m^* \cdot t}{\tau_o}\right)e^{-\frac{m^* \cdot t}{\tau_o}} \cdot \cos(b_x x + \varphi_o)} \qquad (9)$$

The coefficients in the previous equation take the following values:

1) If the opposite faces are free to balance then:

$$\Theta_{ox} = u_{hx} - u_{fx}, \qquad \lim_{t\to\infty} u(t,x) = u_{fx} = \frac{u_{hx} + u_{Cx}}{2}$$

$$m^* = 1, \quad b_x = \frac{\pi}{w \cdot L_x}, \quad w = 1, \quad \varphi_o = 0$$

2) If the opposite faces have constant temperatures (no "t" time exists) then:

$$\Theta_{ox} = u_{hx} - u_{fx}, \quad \lim_{t\to\infty} u(t,0) = u(0,0), \quad \lim_{t\to\infty} u(t,L_x) = u(0,L_x)$$

$$m^* = 0, \quad b_x = \frac{\pi}{w \cdot L_x}, \quad w = 1, \quad \varphi_o = 0$$

3) Only the face x=0 has steady temperature:

$$\Theta_{ox} = u_{Cx} - u_{fx} = u_{Cx} - u_{hx}, \quad \lim_{t\to\infty} u(t,x) = u_{fx} = u_{hx}$$

$$m^* = 1, \quad b_x = \frac{\pi}{w \cdot L_x}, \quad w = 2, \quad \varphi_o = \frac{3\pi}{2}$$

4) Only the face x=L has steady temperature:

$$\Theta_{ox} = u_{hx} - u_{fx} = u_{hx} - u_{Cx}, \quad \lim_{t\to\infty} u(t,x) = u_{fx} = u_{cx}$$

$$m^* = 1, \quad b_x = \frac{\pi}{w \cdot L_x}, \quad w = 2, \quad \varphi_o = 0$$

with similar equations describing thermal stationary waves in the other axes as well. The symbols in the equations are defined as following:

t = time, τ_o = characteristic response time of the material

u = temperature, u_{fx}, u_{fy}, u_{fz} = final temperature in each direction

f_x, f_y, f_z = the difference of a point's temperature from its final temperature

Θ_{ox}, Θ_{oy}, Θ_{oz} = maximum temperature difference from the final temperature

u_{hx}, u_{hy}, u_{hz}, u_{cx}, u_{cy}, u_{cz} = temperatures of hot and cold faces in 3 axes

L_x, L_y, L_z = the thickness of the material in every direction

The total temperature difference of each point from the final value would be the average of the three differences. So, for each point the following would apply:

$$\boldsymbol{f(t,x,y,z) = \frac{f_x(t,x) + f_y(t,y) + f_z(t,z)}{3}} \tag{10}$$

with

$$t \geq 0, \quad 0 \leq x \leq L_x, \quad 0 \leq y \leq L_y, \quad 0 \leq z \leq L_z$$

4. Energy Transfer and Time

4.1. "Thermal discharge" time

It is the characteristic time in Equation 9 when the two opposite sides of the solid are free to equilibrate. So, that

$$c = \frac{\omega}{b} \Rightarrow \sqrt{\frac{k_\sigma}{\rho c_{sp} \tau_o}} = \frac{1}{b\tau_o} \Rightarrow \tau_o = \frac{\rho c_{sp}}{k_\sigma b^2}.$$

If the classical theory of diffusion is valid, then could be assumed that the hot part of the solid is in contact with the intermediate area which is an area of a constant temperature $u(\frac{L}{2})$. According to the Fourier number (F_o), the diffusion in a cuboid would be accomplished when F_o=1 which occurs at time $t_f = \frac{L^2 \rho c_{sp}}{k_\sigma}$ [33].

Thermal Strings are based on the equation: $q(r, t+\tau_o) = -k_\sigma \nabla u(r, t+\tau_o)$. If it is set $t' = t + \tau_o$ then $q(r, t') = -k_\sigma \nabla u(r, t')$ which is the same with the Fourier equation of the thermal diffusion. So, for the Thermal

Strings Theory to be valid, the total time of heat transfer from the hot to the cold part of the solid must be equal to the time of the classical theory.

Therefore, $t_f = \pi^2\tau_o$ (with $b = \frac{\pi}{L}$). Indeed, if is set in the formulas of the function f the time $t = t_f = \pi^2\tau_o$ then :

$$\frac{f(t_f)}{f(0)} = \frac{1+\pi^2}{e^{\pi^2}} = 0.00056 = 0.056\%$$

This shows that the temperature has reached its final value by 99.94% and that the requirement of the two theories calculating equal thermal discharge times is satisfied. If one side has a constant temperature and is in a steady state, then $\tau_o = 4\frac{L^2\rho c_{sp}}{k_\sigma \pi^2}$ (for $b = \frac{\pi}{2L}$). But again would be $t_f = \pi^2\tau_o$ or four times greater.

4.2. Stored and transferred energy

According to the Thermal Strings theory it would be:

$$q + \tau_o \frac{\partial q}{\partial t} = -k_\sigma(\nabla u + \tau_o \frac{\partial}{\partial t}(\nabla u))$$

In one dimension:

$$q + \tau_o \frac{\partial q}{\partial t} = -k_\sigma\left(\frac{\partial u}{\partial x} + \tau_o \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)\right) \Rightarrow q + \tau_o \frac{\partial q}{\partial t} = -k_\sigma\left(\frac{\partial f}{\partial x} + \tau_o \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial x}\right)\right)$$

If in the last equation Equation 9 is implemented for the free boundaries of a string, then

$q_x + \tau_o \dot{q_x} = k_\sigma\, b_x \Theta_{ox} e^{-\frac{t}{\tau_o}} \sin(b_x x) = Z(x)e^{-\frac{t}{\tau_o}}$ with $Z(x) = k_\sigma\, b_x \Theta_{ox} \sin(bx)$

The solution of the above differential equation is $q_x = c_1 e^{-\frac{t}{\tau_o}} + \frac{Z(x)t}{\tau_o} e^{-\frac{t}{\tau_o}}$

Because, there is no flux at the beginning, $q_x(0) = 0 \Rightarrow c_1 = 0$ then

$$q_x = \frac{Z(x)t}{\tau_o} e^{-\frac{t}{\tau_o}} \quad (11)$$

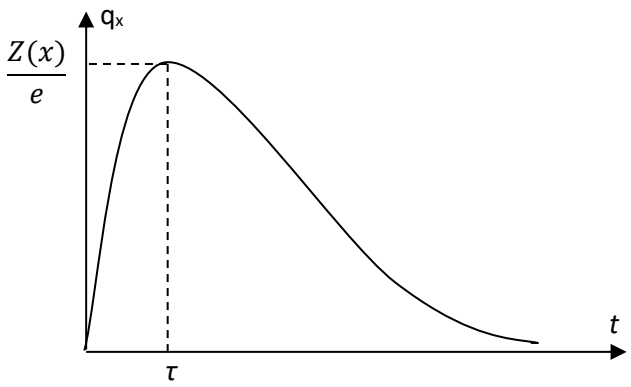

Figure 3: Heat transfer/Surface/time (W/m^2)

The heat flow versus time is shown in Figure 3. Along a string of length $0 \le x \le \frac{L}{2}$ the heat that is transferred in the form of thermal waves would be equal to $\int_0^\infty q_x\, dt = \int_0^\infty \frac{Z(x)t}{\tau_o} e^{-\frac{t}{\tau_o}} dt = Z(x)\tau_o$ and the total heat to be transferred from one half of the solid to the other half results from the difference of the stored heat at the beginning and at the boundary. So, it is $Q = \int_0^\infty q_x A\, dt = \rho c_{sp} A \frac{1}{b_x}\Theta_{ox} = Z(x)A\tau_o$. If at $Z(x)$ it is set $x = \frac{L}{2}$

then again it is concluded the value of the τ_o the same as before: $\tau_o = \frac{\rho c_{sp}}{k_\sigma b^2}$. If it is set the time $t_f = \pi^2 \tau_o$ then in Equation 11 $\frac{q(t_f)}{q(0)} = \frac{\pi^2}{e^{\pi^2}} = 0.00051 = 0.051\% \approx 0$ which means that the energy transfer from the warm to the cold part stops at the same time. So, based on the Thermal Strings Model it could successfully calculate the heat transfer for thermal discharge up to the point of thermal equilibrium.

Moreover, there is a total amount of heat energy passing and leaving from every single point on x axis during the phenomenon of conduction. This amount consists of a) the heat that is arriving to this point from the preceding points which are getting cold and b) its own internal energy that is lost while the point is getting cold. The transferred energy from every point is described by the integral $Q_{point}(x,t) = \int q(x,t)dt = -Z(x) \cdot (t + \tau_o)e^{-\frac{t}{\tau_o}} + c(x)$.

At the beginning (t=0) it is: $Q_{point}(x,0) = 0 \Rightarrow c(x) = Z(x) \cdot \tau_o$. So, it is obtained the transferred energy through a whole cross section as:

$$Q(x,t) = Z(x) \cdot A \cdot \tau_o \cdot \left(1 - e^{-\frac{t}{\tau_o}}\right) - Z(x) \cdot A \cdot t \cdot e^{-\frac{t}{\tau_o}} \quad (12)$$

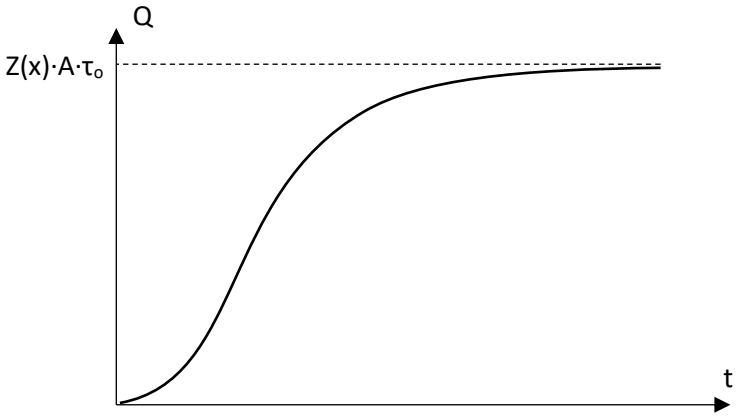

Figure 4: Heat transfer/Surface (J/m^2)

The initial storage energy at the hot half part of the body will be lost and will be transferred to the cold part. This energy is

$$\Delta U_\Omega = \int_0^{\frac{L}{2}} \rho c_{sp} A \frac{1}{b_x} \Theta_{ox} \cos(b_x x)\left(1 + \frac{t}{\tau_o}\right) e^{-\frac{t}{\tau_o}} dx \overset{t=0}{\Longrightarrow} \Delta U_\Omega = \rho c_{sp} A \frac{1}{b_x} \Theta_{ox}$$

According to Equation 12, from the middle area at $x = \frac{L}{2}$ the transferred energy would be

$$Q(x,t) = Q\left(\frac{L}{2}, \infty\right) = k_\sigma\, b_x \Theta_{ox} A \tau_o$$

But, the wave number is given by $b_x = \frac{\omega_o}{c} = \frac{1}{\tau_o c} \Rightarrow c^2 = \frac{1}{b_x^2 \tau_o^2} \Rightarrow \frac{k_\sigma}{\rho c_{sp} \tau_o} = \frac{1}{b_x^2 \tau_o^2} \Rightarrow \rho c_{sp} = k_\sigma b_x^2 \tau_o$

so, $\Delta U_\Omega = \rho c_{sp} A \frac{1}{b_x} \Theta_{ox} = k_\sigma\, b_x \Theta_{ox} A \tau_o = Q\left(\frac{L}{2}, \infty\right)$. This is a proof that the lost energy from the hot part would pass through the middle area and arrive at the cold part. It is like the middle area acting as a "tunnel". Its temperature (and its internal energy) remains steady and the transferred energy passes through it without affecting its temperature. With time, the heat which is travelling through every cross section approaches to an upper limit ($Z(x)\cdot A\cdot \tau_o$) depending on its x position because it allows the whole amount of energy from its preceding points.

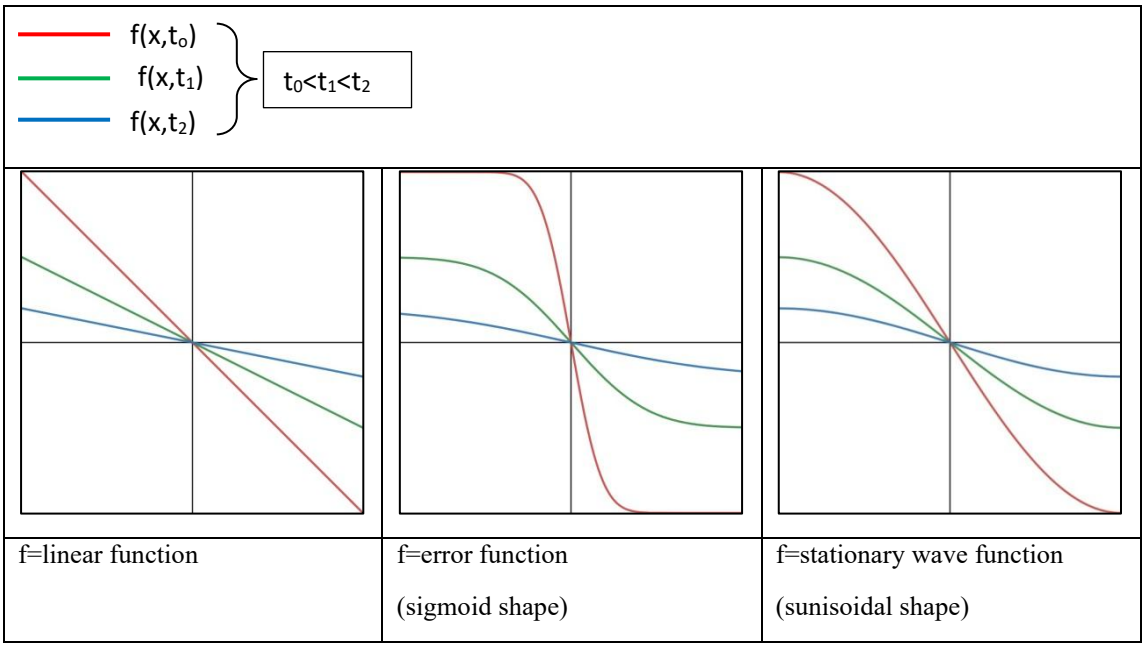

Figure 5: Temperature distributions and f-functions (time-depended)

In Figure 5 we can see three different temperature distributions in a "half-hot/half-cold" cuboid. The first one is a linear distribution, the second has sigmoid shape, and the third has sinusoidal shape. All three of them turn to be flat line after approaching equilibrium and the internal energy is proportional to the temperature. The cuboid is insulated, so the internal energy has to be constant and the calculated energy contained in the solid (volume Ω) is derived from the integral $U_\Omega = C\int_0^L u\, d\Omega = constant\ (time\ independed)$. In all three cases this energy remains constant from the beginning until the end of the heat transfer. In fact, the different shape of each distribution yields to different amounts of stored energy into the hot part. But there is a minus identical amount into the cold part that is why the total stored energy remains the same. After comparing a linear and a sinusoidal distribution at the same cuboid and with the same boundary and initial conditions, it was found 27% more energy into the hot part and less 27% energy into the cold part (Figure 6).

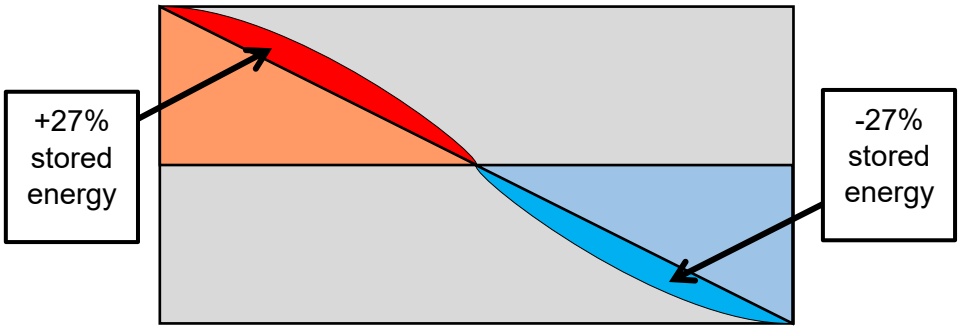

Figure 6: Initial temperature distribution and energy storage (linear and sinusoidal shape)

Another useful calculation that proves the correctness of the thermal string model has to deal with the case where the opposite sides of a homogeneous solid are in steady state. Then, energy will be constantly transferred from one side to the other. To find the total energy transferred per unit time of the material (for $0 \le x < \frac{L}{2}$) must calculate the integral $\dot{Q}_x = \int_0^{\frac{L}{2}} q_x A dx = k_\sigma A \Theta_{ox}$. It is set $q = -k_\sigma \frac{\partial u}{\partial x}$ because $\frac{\partial u}{\partial t} = 0$ and because u=constant and q=constant therefore $\dot{q} = 0$. According to the Mean Value Theorem a constant power $\overline{\dot{Q}_x}$ exists such that

$\int_0^{\frac{L}{2}} q_x A dx = \overline{\dot{Q}_x}(\frac{L}{2} - 0)$ and therefore $\overline{\dot{Q}_x} = \frac{k_\sigma A \Theta_{ox}}{L/2}$. This value is the same that results from Fourier's diffusion equation:

$$\frac{Q}{t} = k_\sigma A \frac{\Delta u}{d} = k_\sigma A \frac{\Delta f}{L/2} = k_\sigma A \frac{(\Theta_o - 0)}{L/2} = \frac{k_\sigma A \Theta_o}{L/2}$$

So in general, the energy would be $Q = k_\sigma A \frac{\Delta u}{d} t = \frac{1}{R} A \Delta u \cdot t$. The R here is the thermal resistance of the material with units (m^2K/W). If 1/R is replaced by the material's U_{VALUE} (U_V=1/R) the power on a string would be $q_x = \frac{1}{R} \Delta(f)$. But due to

$\Delta f = f_2 - f_1 = \frac{f_x(L_x)+f_y(y)+f_z(z)}{3} - \frac{f_x(0)+f_y(y)+f_z(z)}{3} = \frac{\Delta f_x}{3} = \frac{2\Theta_{ox}}{3}$, it is finally concluded that $q_x = \frac{1}{R}\frac{2\Theta_{ox}}{3}$. The thermal power through a surface with area A_{yz} will be $\frac{Q}{t} = \frac{1}{R} A_{yz} \frac{2\Theta_{ox}}{3} = \frac{1}{R} L_y L_z \frac{2\Theta_{ox}}{3}$. Generally, in a solid when temperature fluctuations occur by Δ(temperature)=$2\Theta_{ox}$ on one side, then on the other side the fluctuations have an amplitude equal to Δ(temperature)/3 because the other sides of the solid that keep their temperature constant also contribute to the total temperature (Figure 7).

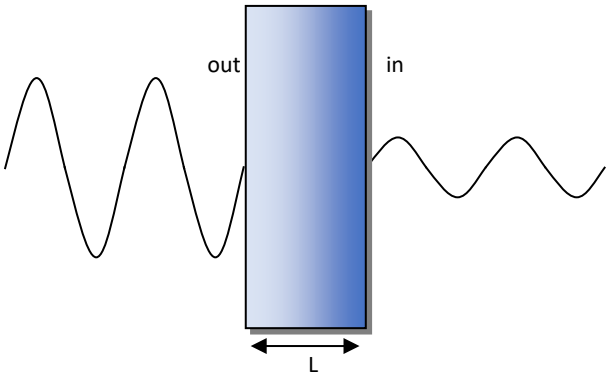

Figure 7: Temperature variations outside and inside of a wall

5. Heat conduction in masonry materials

Let us compare three kinds of wall-materials such as brick, stone and extruded polystyrene. Brick is a common material in building walls, stone is a material at the old buildings and extruded polystyrene is an insulation material also used in new buildings. Their properties are given in TABLE 1. They have a thickness of 20cm and their surface area is $1m^2$. It is set a temperature difference Δu=1°C between the two faces of each wall.

According to current calculation by implementing the Thermal Strings model, it is found the time required to reach equilibrium so the two faces finally have the same temperature. This time to equilibrate is the same with the one calculated by the Fourier's model. It is, also, found the amount of heat which is transferred from the hot side to the cold side of the wall. This amount is 27% greater than the one in the classical model. But the total internal energy into the wall is the same with the one at the classical model.

In order to keep the temperature difference permanently at Δu=1°C between the two sides, we have to supply energy per second. This power is the same with the one calculated with the classical model.

The results of the calculations are given in TABLE 1:

TABLE 1: Comparison between 3 materials.

Material	Brick	Stone	Extruded polystyrene
Thickness L (m)	0.2	0.2	0.2
Thermal conductivity k (W/m·K)	0.711	2.3	0.038

Density ρ (kg/m^3)	2000	2600	50
Specific heat capacity c_{sp} (J/kg·m)	837	1000	1450
Time for equilibrium t (h)	26.16	12.56	21.19
Heat transformation Q (J)	17761	27586	769
Power needed for steady state $\bar{\dot{Q}}$ (W)	1.185	3.83	0.063

Table 1 showed that the time required to equilibrate is not depended on the temperature difference. The stone will reach equilibrium in 12.5 hours when the brick would need more than 26 hours. So, the brick has greater thermal resistance than the stone. Stone houses have much thicker walls and the thickness is what greatly changes the heat transfer time whereas doubling of the thickness quadruples the time (time depends on L^2).

If extruded polystyrene with the same thickness (20cm) is used then the propagation time of a pulse would be 21.2 hours. The advantage of extruded polystyrene (and other similar insulating materials) is not only the propagation time of temperature disturbances but also the energy required so that these changes not to have any effect. So, when the temperature outside a house drops, the indoor temperature could be kept high by supplying small amount of energy.

If one side of the wall is heated in order to transfer the energy to the whole solid body, it would require four times more time than if it was free to equilibrate.

The energy per unit time needed to heat or cool one side of a wall, so that it has a permanent fixed difference of temperature of 1°C from the outside environment, is about 20 times less if the wall is made of insulating material rather than bricks. This means that with 1/20 of the insulation thickness in one layer combined with a brick it can halve the required energy.

The air, as an insulating material, has even better properties because it is 30 times more effective than a brick. In case of humidity inside the wall, then the air loses its effectiveness, because it needs 25.5 times more power that we have to spend to heat a wall by 1^{o}C per unit area.

6. Conclusion

The Thermal Strings model is an attempt to compromise the Fourier and the DPL models of heat conduction considering that every material has a consistent behavior not depending on the duration of the phenomena.

List of the basic assumptions of the Thermal Strings model:

- Heat is conducted by a finite speed.
- There are two lag times of the solid's response to temperature changes describing a) the heat flow delay and b) the delay due to thermal inertia. The two times are equal.
- The heat absorption/storage and the heat transfer are two phenomena working simultaneously and competitively.
- Each axis of the solid is a thermal string. The shape and the form of the string determine the distribution of temperature inside the solid and explain the existence of points with fixed temperature (at the boundaries or in the middle of each axis).
- Due to thermal fluctuations, thermal stationary waves appear in the string.
- It is assumed from initial conditions what would be the final temperature of the solid (target temperature). There are subcases defined by the boundary conditions.

Main findings and results of the Thermal Strings model:

- Each point of the solid has a temperature difference f from its final temperature. This difference f decreases over time as the point tends to its predetermined final temperature. The same occurs if it is used the non-dimensional temperature.

- The temperature behavior of each point is equivalent to damped harmonic oscillation whereas oscillation damping is critical. Each line is a thermal string that pulsates thermally with decreasing temperature oscillations.
- Every point belongs to three axes x,y,z and three stationary waves. The total temperature difference f_{xyz} of a point, from its final, is the average of f_x, f_y, f_z. Inside the material space there are points with a total $f_{xyz}=0$ (points with fixed temperature). These points are equivalent to nodes of the thermal strings.
- The stationary thermal waves phenomenon led to the quantisation of the temperature distribution.
- For a free-to-equilibrate solid, the necessary period is about 10 times greater than the characteristic time of the material ($t_f = \pi^2 \tau_o$). For a non-free-boundary solid the total time is 4 times greater than before, so the relaxation time is increased. For an all-boundaries-steady-temperature solid, energy must be supplied continuously to keep the difference between the hot and the cold areas.
- The molecules form a chain and do not change temperature independently of each other as the heat simultaneously is absorbed and transferred from each particle. Heat's absorption and storage causes delay of heat propagation.

Validation (comparison with previous results):

- The Thermal Strings model has been verified against both the diffusion theory and the DPL theory.
- It is an expansion of DPL and provides an analytical solution for every kind of solid, any dimension, for any time and even for solids with layered materials (inhomogeneous).
- The results from this model are in full agreement with the classical theory.
- Although, it is based on the wave theory, gives the same results against the Fourier's theory compared to:

1. the duration of heat transfer that takes place between two solids in contact (difference from classical theory by 0.056%)
2. the heat loads required to keep non-equilibrium conditions (difference from classical theory by 0.051%)
3. the Mean Value Theorem for the thermal power function.

Concluding remark is that the main advantage of the Thermal Strings model is that can be applied to computational calculations to give results in short time for different parameters such as relaxation time, transferred energy, heat load per time unit as long as the boundary conditions are known with a high degree of accuracy. This is possible even with masonry materials that consist of multiple layers.

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