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Generalized New Linear-Exponential Distribution

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Abstract:

This distribution has been constructed for improvement in statistical modelling while using New Linear-exponential Distribution (NLED) just by adding an additional parameter to its probability density function. The proposed distribution is named as "Generalized New Linear-exponential Distribution". Its important characteristics have been defined, derived and constructed in well and systematic manner. It has been verified by using some secondary data that the proposed distribution is a better alternative of NLED for statistical modelling.

Keywords: Probability distribution, Distribution, New Linear-exponential distribution, Estimation, Moments, Chi-square goodness of fit test.

1.0 Introduction:

Here, we are going to introduce a two-parameter continuous distribution which is a generalized case of NLED [1] and the probability density function of NLED was obtained by Sah (2022) and given as

$$f(u,\beta)=\frac{\beta^2}{(1+\pi\beta)}(\pi+u)e^{-\beta u} \tag{1}$$

Where, $u>0$ and $\beta>0$.

The expression (1) was obtained by the help of a linear function ($(\pi+u)$ and an exponential function $e^{-\beta u}$. In the proposed distribution, we use a linear function $(\pi\tau+u)$ and an exponential function $e^{-\beta u}$ to form density function of Generalised New Linear-exponential distribution. It is abbreviated as GNLED. NLED is a particular case of GNLED at $\tau=1$.When we apply chi-square goodness of fit test to some survival time data, it has been observed that GNLED is a better alternative to NLED because of introducing an additional parameter to the density function of the proposed distribution.

Adding another parameter to the previously derived distribution is also an art and in which positioning the additional parameter is also a big challenge. In the linear function $(\pi\tau+u)$ used for the construction of GNLED, τ is the additional parameter which plays a very important role for improving in results. If we add an additional parameter to Premium Linear-exponential distribution [2], New Quadratic-exponential distribution [3] and Modified Mishra distribution [4], we will get better alternatives of these distributions. The references [5] and [6] are very helpful for preparing this distribution.

Presentation of this paper has been classified under following headings:

1.0 Introduction
2.0 Results
3.0 Applications
4.0 Conclusions

2.1 Probability Density Function ($f(u;\theta,\beta)$), Moment Generating Function (M_u^t) and Distribution function ($F(u)$) of GNLED:

Probability density function of variable u such that $u>0$ with parameters β and θ can be defined as

$$f(u;\theta,\beta)=\frac{\beta^2}{(1+\pi\beta\theta)}(\pi\theta+u)e^{-\beta u} \tag{2}$$

Where, $\beta>0$; $(1+\pi\theta\beta)>0$ **.** NLED is a particular case of GNLED (2) at $\theta=1$.

Figure-1 Probability Mass Function of GNLED at $\theta=5$ *and* $\beta=0.1,0.2,0.3,0.4$

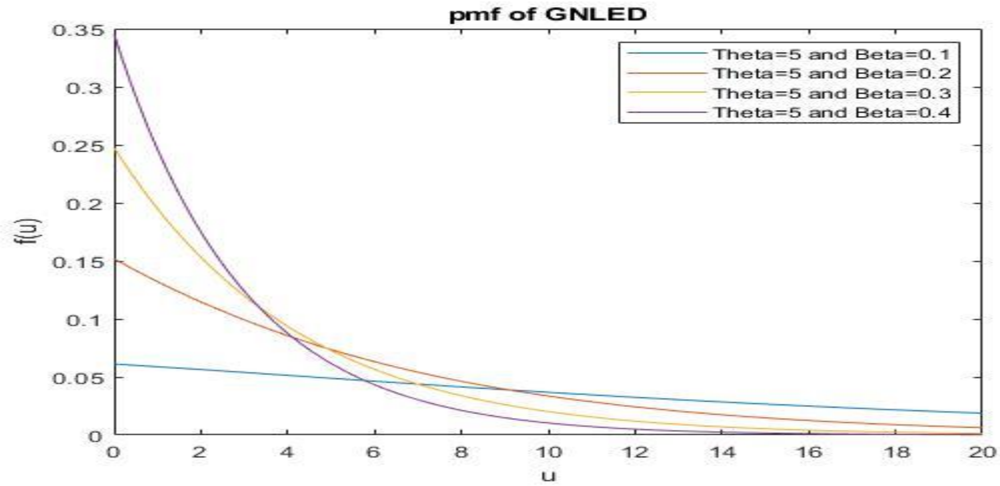
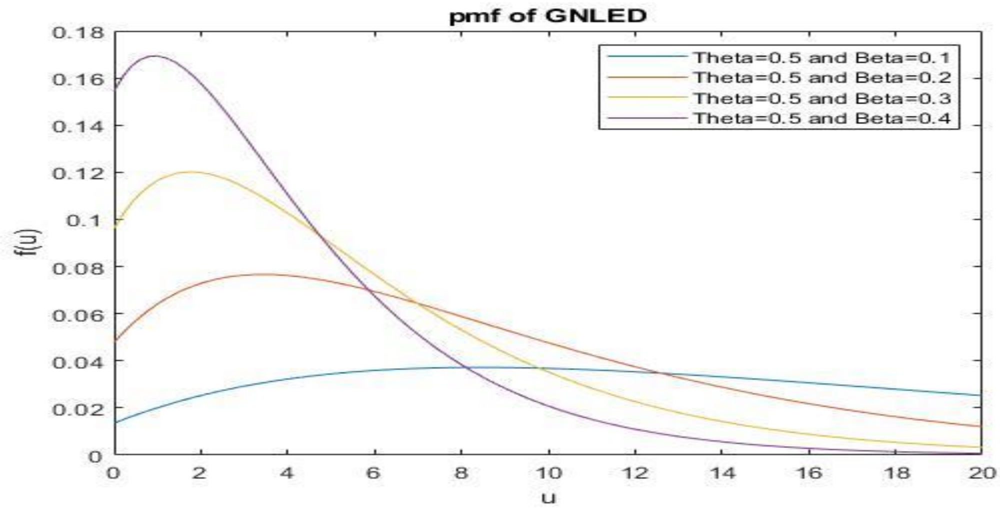


Figure-2 Probability Mass Function of GNLED at $\theta=0.5$ *and* $\beta=0.1,0.2,0.3,0.4$



Moment generating function of GNLED can be obtained as and given in the expression (3).

$$M_u^t=\frac{\beta^2}{(1+\pi\theta\beta)}\int_0^\infty(\pi\theta+u)e^{-\beta u}du$$

Or, $M_u^t=\frac{\beta^2}{(1+\pi\theta\beta)}\left[\pi\theta\int_0^\infty e^{-(\beta-t)u}du+\int_0^\infty ue^{-(\beta-t)u}du\right]=\frac{\beta^2}{(1+\pi\theta\beta)}\left[\frac{\pi\theta}{(\beta-t)}+\frac{1}{(\beta-t)^2}\right]$

Or, $M_u^t=\frac{\beta^2}{(1+\pi\theta\beta)}\left[\frac{\pi\theta(\beta-t)+1}{(\beta-t)^2}\right]$ (3)

Distribution function of GNLED can be derived as follows and given by the expression (4).

$$F(u) = P(U \le u) = \int_0^u f(u)du = \frac{\beta^2}{(1+\pi\beta\theta)}\int_0^u (\pi\theta + u)e^{-\beta u}du = 1 - \frac{(1+\pi\beta\theta+\beta u)}{(1+\pi\beta\theta)}e^{-\beta u} \tag{4}$$

Figure-3 Probability Distribution Function of GNLED at $\theta = 0.5$ *and* $\beta = 0.1, 0.2, 0.3, 0.4$

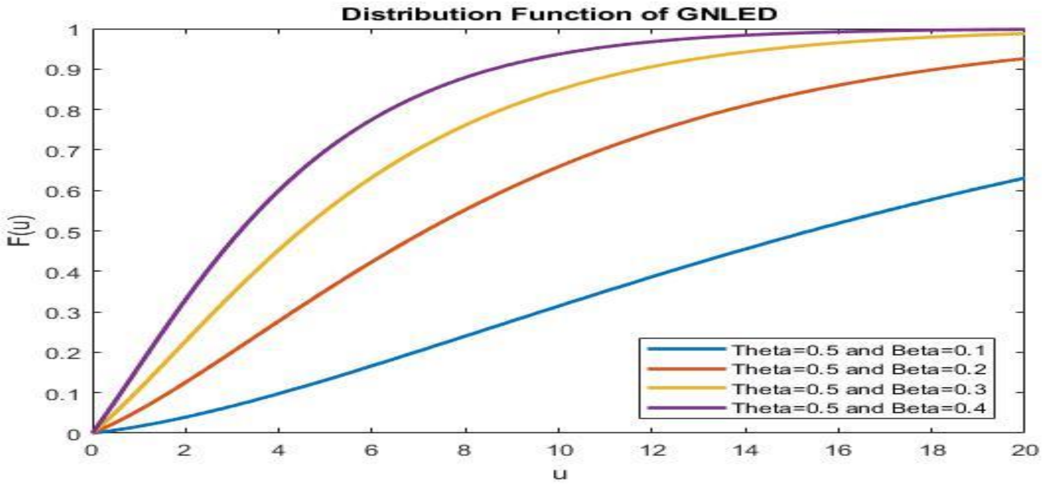
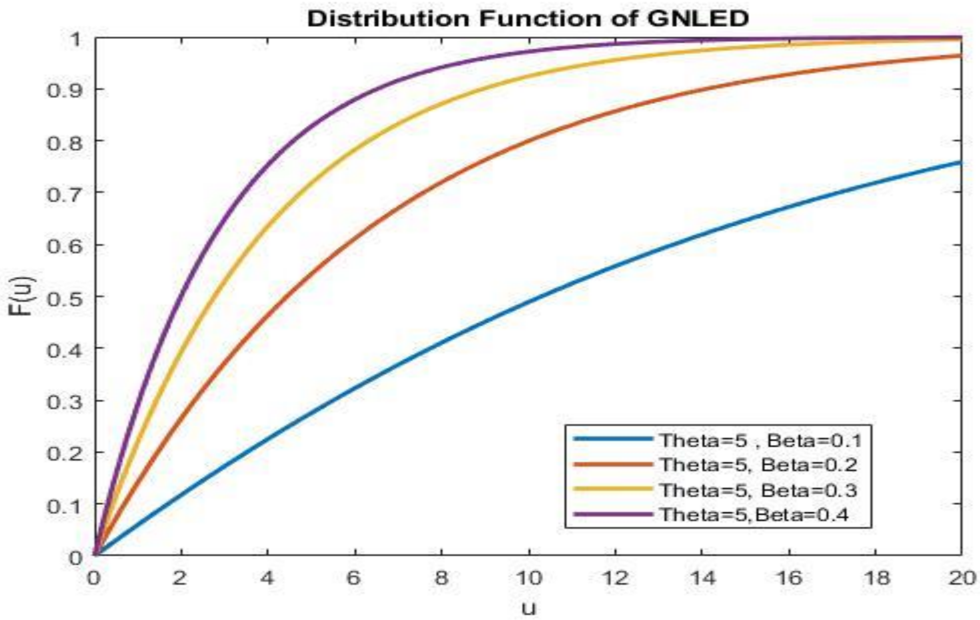


Figure-4 Probability Distribution Function of GNLED at $\theta = 5$ *and* $\beta = 0.1, 0.2, 0.3, 0.4$



2.2 Moments about the origin (μ_r') and Central Moments of GNLED (μ_r):

The r[th] moment about the origin, the first four moments about the origin and the first four central moments can be derived as follows and given in the equations (5) to (13) in order respectively.

$$\mu_r' = \frac{\beta^2}{(1+\pi\beta\theta)}\int_0^\infty u^r(\pi\theta+u)e^{-\beta u}du = \frac{\beta^2}{(1+\pi\beta\theta)}\left[\pi\theta\int_0^\infty u^r e^{-\beta u}du + \int_0^\infty u^{r+1}e^{-\beta u}du\right]$$

Or, $$\mu_r' = \frac{r!}{\beta^r}\left[\frac{1+r+\pi\theta\beta}{(1+\pi\theta\beta)}\right] \tag{5}$$

$$\mu_1' = \frac{1!}{\beta^1}\left[\frac{(2+\pi\theta\beta)}{(1+\pi\theta\beta)}\right] \tag{6}$$

$$\mu_2' = \frac{2!}{\beta^2}\left[\frac{(3+\pi\theta\beta)}{(1+\pi\theta\beta)}\right] \tag{7}$$

$$\mu_3' = \frac{3!}{\beta^3}\left[\frac{(4+\pi\theta\beta)}{(1+\pi\theta\beta)}\right] \tag{8}$$

$$\mu_4' = \frac{4!}{\beta^4}\left[\frac{(5+\pi\theta\beta)}{(1+\pi\theta\beta)}\right] \tag{9}$$

$$\mu_1 = 0 \tag{10}$$

$$\mu_2 = \frac{2!}{\beta^2}\left[\frac{(3+\pi\theta\beta)}{(1+\pi\theta\beta)}\right] - \left\{\frac{1!}{\beta^1}\left[\frac{(2+\pi\theta\beta)}{(1+\pi\theta\beta)}\right]\right\}^2$$

$$\text{Or, } \mu_2 = \left[\frac{2+4\pi\theta\beta+(\pi\beta\theta)^2}{(\beta^2+2\pi\theta\beta^3+\pi^2\theta^2\beta^4)}\right] \tag{11}$$

$$\mu_3 = \frac{3!}{\beta^3}\left[\frac{(4+\pi\theta\beta)}{(1+\pi\theta\beta)}\right] - 3\frac{2!}{\beta^2}\left[\frac{(3+\pi\theta\beta)}{(1+\pi\theta\beta)}\right]\frac{1!}{\beta^1}\left[\frac{(2+\pi\theta\beta)}{(1+\pi\theta\beta)}\right] - 2\left\{\frac{1!}{\beta^1}\left[\frac{(2+\pi\theta\beta)}{(1+\pi\theta\beta)}\right]\right\}^3$$

$$\text{Or, } \mu_3 = \left[\frac{4+12(\pi\theta\beta)+12(\pi\beta\theta)^2+2(\pi\beta\theta)^3}{(\beta(1+\pi\theta\beta))^3}\right] \tag{12}$$

$$\mu_4 = \frac{4!}{\beta^4}\left[\frac{(5+\pi\theta\beta)}{(1+\pi\theta\beta)}\right] - 4\frac{3!}{\beta^3}\left[\frac{(4+\pi\theta\beta)}{(1+\pi\theta\beta)}\right]\frac{1!}{\beta^1}\left[\frac{(2+\pi\theta\beta)}{(1+\pi\theta\beta)}\right]$$

$$+6\frac{2!}{\beta^2}\left[\frac{(3+\pi\theta\beta)}{(1+\pi\theta\beta)}\right]\left\{\frac{1!}{\beta^1}\left[\frac{(2+\pi\theta\beta)}{(1+\pi\theta\beta)}\right]\right\}^2 - 3\left\{\frac{1!}{\beta^1}\left[\frac{(2+\pi\theta\beta)}{(1+\pi\theta\beta)}\right]\right\}^4$$

$$\mu_4 = \left[\frac{\left\{24+96(\pi\theta\beta)+132(\pi\beta\theta)^2+72(\pi\beta\theta)^3+9(\pi\beta\theta)^4\right\}}{\left\{\beta(1+\pi\theta\beta\right\}^4}\right] \tag{13}$$

2.3 Nature of GNLED According to Dispersion, Skewness and Kurtosis:

(A) *Index of Dispersion (I):*

$$I = \frac{\mu_2}{\mu_1'} = \frac{(2+4\pi\theta\beta+(\pi\beta\theta)^2)}{\beta(1+\pi\theta\beta)(2+\pi\theta\beta)} \tag{14}$$

This distribution will be Over-dispersed, Equi-dispersed and Under-dispersed when $I>1, I=1$ and $I<1$ respectively.

(B) *Coefficient of Skewness (γ_1):*

$$\gamma_1 = \frac{\left\{4+12(\pi\theta\beta)+12(\pi\beta\theta)^2+2(\pi\beta\theta)^3\right\}}{\left\{2+4\pi\theta\beta+(\pi\beta\theta)^2\right\}^{3/2}} \tag{15}$$

It lies between $(\sqrt{2})$ and ∞. Hence, it is positively skewed.

(C) *Coefficient of Kurtosis (β_2):*

$$\beta_2 = \frac{\left\{24+96(\pi\theta\beta)+132(\pi\beta\theta)^2+72(\pi\beta\theta)^3+9(\pi\beta\theta)^4\right\}}{\left\{2+4\pi\theta\beta+(\pi\beta\theta)^2\right\}^2} \tag{16}$$

It lies between 6 and ∞ .Hence, it is Leptokurtic by size.

2.4 Estimation of Parameters of GNLED:

This distribution has two parameters β and θ which can be estimated by using the expressions (17) and (18). Using the expression (6), we get

$\mu_1'\beta(\theta+\pi\beta)=(2\theta+\pi\beta)$

Or, $\bar{u}\beta\theta-2\theta=\pi\beta-\bar{u}\pi\beta^2$

Or, $\theta(\bar{u}\beta-2)=(\pi\beta-\bar{u}\pi\beta^2)$

Or, $\hat{\theta}=\dfrac{(\pi\beta-\bar{u}\pi\beta^2)}{(\bar{u}\beta-2)}$ (17)

Using the expression (7) and substituting the value of θ , we get

$\mu_2'\beta^2-4\mu_1'\beta+2=0$

Or, $s_2\beta^2-4\bar{u}\beta+2=0$

It is a quadratic equation in terms of β , solving it we get an estimator of β as $\hat{\beta}$

$$\hat{\beta}=\frac{4\bar{u}\pm\sqrt{\{(4\bar{u})^2-4(s_2)(2)\}}}{2s_2}$$

$$\hat{\beta}=\frac{2\bar{u}\pm\sqrt{\{(4\bar{u}^2-2s_2)\}}}{s_2} \tag{18}$$

Where s_2 is the second sample moment about the origin of GNLED.

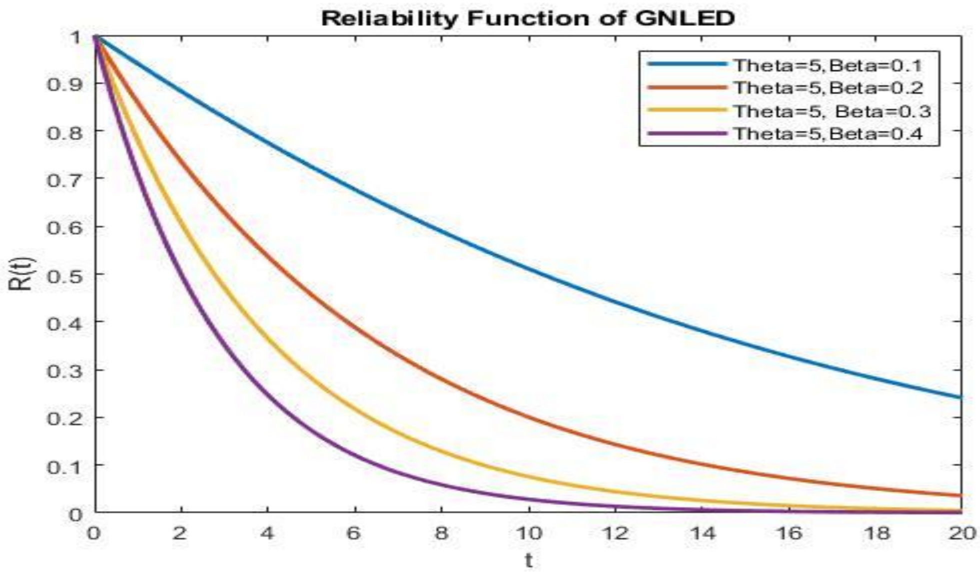
2.5 Hazard Rate Function ($h(u=t)$), Reliability Function ($R(u=t)$) and Mean Residual Life Function ($m(u)$): These functions are derived and given by the expression (19), (20) and (21) respectively.

$$h(u=t)=\frac{\beta^2(\pi\theta+t)}{(1+\pi\beta\theta+\beta t)} \tag{19}$$

$$h(u=0)=\frac{\beta^2(\pi\theta)}{(1+\pi\beta\theta)}$$

$$R(u=t)=\frac{(1+\pi\beta\theta+\beta t)}{(1+\pi\beta\theta)}e^{-\beta t} \tag{20}$$

Figure-4 Reliability Function of GNLED at $\theta=5$ *and* $\beta=0.1,0.2,0.3,0.4$



$$m(u)=\frac{(2+\pi\beta\theta+\beta u)}{\beta(1+\pi\beta\theta+\beta u)} \tag{21}$$

$m(u=0)=\dfrac{(2+\pi\beta\theta)}{\beta(1+\pi\beta\theta)}$. It is the mean of GNLED.

3.0 Applications of GNLED:

This distribution is better than NLED for statistical modelling of survival life time data, failure rate of machine in guarantee periods data. Here, we use Chi-square goodness of fit test to the following example.
Example (1):

Table-1

Survival times (in days) of guinea pigs infected with virulent tubercle bacilli, reported by Bzerkedal [7]

Survival time (in days)	0-80	80-160	160=240	240-320	320-400	400-480	480-560
Observed Frequency	8	30	18	8	4	3	1

Table-2

Tabulation of Theoretical Frequencies, Degrees of Freedom, Values of Chi-square and P-values by using NLED and GNLED.

Survival Time (in days)	Observed Frequency	Expected Frequency	
		NLED	GNLED
0-80	8	16.4	10.7
80-160	30	22.6	26.9
160-240	18	15.3	17.7
240-320	8	9.0	9.2
320-400	4	4.9	4.3
400-480	3	2.5	1.9
480-560	1	2.3	1.3
Total	72.0	72.0	72.0
$\bar{u}$ =181.11111	$\mu_2' = 43911.11111$	-	-
$\hat{\theta}$	-	-	-6.392776012
$\hat{\beta}$	-	0.010860	0.01299232434
$\mathrm{d}.f.$	-	3	2
$\chi^2_{d.f.}$	-	8.43	1.07
$P-Value$	-	0.0379	0.5857

4.0 Conclusion:

From the second table, we can see that the P-value due to GNLED is greater than NLED, and therefore we can say that GNLED seems more suitable than NLED for Statistical modelling of survival time data.

Conflict of Interest:

The authors of this paper have selflessly focused on strengthening the theory of continuous probability distribution basics and have no other ill intensions in mind.

Acknowledgement:

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Data Sources: The data used in this paper are secondary in nature and references are provided.

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