International Journal of Multiphysics
Volume 18, No. 4, 2024
ISSN: 1750-9548

Iterative Methods in Hadamard Manifolds

Mehdi Rabani[1], Ali Parsian[2]*, Esmaeil Nazari[3]

[1]M.Sc. Student in Pure Mathematics (Geometry), Department of Pure Mathematics, Faculty of Mathematical Sciences, Tafresh University, Tafresh, Markazi Province, Iran
[2]Associate Professor, Department of Pure Mathematics, Faculty of Mathematical Sciences, Tafresh University, Tafresh, Markazi Province, Iran
[3]Assistant Professor, Department of Pure Mathematics, Faculty of Mathematical Sciences, Tafresh University, Tafresh, Markazi Province, Iran.
*Corresponding author

Abstract

This study presents several theorems of Hadamard's manifolds, initially introducing concepts of differential manifolds, including metrics and geodesics, and then examining concepts that determine Hadamard's manifolds. The study later concerns Halpern and Mann's iterations of non-expansive mappings on Hadamard's manifolds, citing a numerical example to present how these methods can be applied, particularly concerning the methods of Halpern and Mann's iterations. Additionally, several applications of convergence results for the origin point algorithm and Picard's iterations are mentioned. The first case concerning a minimization problem is practiced by applying the convergence results on a generated minimal problem by finding a saddle point. This is followed by solving a variational inequality.

Keywords: Hadamard manifold, Halpern algorithm, non-expansive mapping, minimization problems, inequality.

Introduction

This article investigates mathematical constructs on monotone vector fields and non-expansive mappings in specific geometric spaces such as Hadamard's manifolds and Hilbert's spaces, initially introducing and then analyzing fixed point approximations and monotonicity theorems within Hadamard's manifolds [1-5]. The article later examines the concept of resolvent, defined by Inamiya and H. Okochi, in Hilbert's manifold space, along with the concept's well-defined conditions while analyzing the symmetrical behavior of solutions based on Yosida sort characteristics [6-9]. Also, a number of theorems concerning the existence and uniqueness of a solution under boundary conditions are presented, with the fixed-point theorem in Hadamard's manifolds and the concept of firm non-expansive mapping being presented as defined by Goebel and Reich [10] in the Hilber Sphere. The characteristics of these constructs lead to a robust bond between monotone vector fields and firm non-expansive mappings through the concept of resolvent. Furthermore, the concept of complement in vector fields is used to help prove the relationship between monotonicity and the class of pseudo-contraction operators, demonstrated by Reich and Shafrir [11] in the hyperbolic space.

Then, the study examines how various algorithms converge in non-expansive mappings, especially how the Picard iteration method converges in firm non-expansive mappings. On the same line, an approximation method is also presented, as a numerical example of Halpern and Mann's iterations is cited to explain how these methods practically function. In sum, theoretical results of minimization problems, minimal problems, and variational inequalities are utilized to demonstrate wider theoretical applications.

Hadamard's Manifolds

As a concept, local curvature in the Riemannian manifold plays a major role in geometry development. Presented by Rieman as a natural generalization of Gaussian curvature of surfaces, this concept measures the value a Riemannian manifold deviates from being Euclidean. In subsequent years, a simpler equation was offered by Christoffel using the Levi-Civita connection.

Definition: A complete, simply connected Riemannian manifold with non-positive local curvature is called a Hadamard manifold.

The rest of this section always assumes that M is the next Hadamard manifold. The famous result below will be key for this section which can be found in [[12] p. 221 of Theorem 4.1.].

Proposition: Suppose $x \in M$. In this case $exp_x: T_x \to M$ is a diffeomorphism, and for both points $x, y \in M$ there is a single normal geodesic that connects x to y, which is also a minimal geodesic.

This proposition implies a diffeomorphic to the Euclidian space R^m. Therefore, M represents a similar topology and a differential structure like R^m. Moreover, Hadamard's manifolds and Euclidean spaces share similar geometric features.

The following proposition explains one of the main characteristics of the Hadamard manifolds, adopted from [[12], p. 223, Prop. 5.4.]. Let's remember that a geodesic triangle $\Delta(x_1, x_2, x_3)$ of a Riemannian manifold represents a set composed of three points and three minimal geodesic that connects these points in a pairwise manner.

Proposition: Suppose $\Delta(x_1, x_2, x_3)$ is a geodesic triangle in M. Suppose for every i=1, 2, 3 at modulo with $\gamma_x: [0, l_i] \to M$ there is a geodesic connecting x_i to x_{i+1}.

$$l_i = l(\gamma_i), \alpha_i := < \gamma_i'(0), -\gamma_{i-1}'(l_{i-1}') >$$

$$\alpha_1 + \alpha_2 + \alpha_3 \leq \pi$$

$$L_i^2 + L_{i+1}^2 - 2L_iL_{i+1}\cos\alpha_{i+1} \leq L_{i-1}^2$$

The above inequality is used to provide the following:

$$d^2(x_i, x_{i+1}) + d^2(x_{i+1}, x_{i+2}) - 2 < exp_{x_{i+1}}^{-1} x_i exp_{x_{i+1}}^{-1} x_{i+2} > \leq d^2(x_{i-1}, x_i)$$

Firm Non-expansive Mappings

The firm non-expansive concept was already defined in a Banach space [13] [14] and the Hilbert sphere with a hyperbolic metric [10]. The following analyses indicate that in Hadamard manifolds, this class of mappings, the most famous in Hilbert spaces, share similar features.

Definition: The $T: CM \to M$ mapping is given. T is said to be firmly non-expansive if for every $x, y \in M$, the function $\theta: [1] + [0, \infty]$, defined with:

$$\theta(t) = d(\gamma_1(t), \gamma_2(t))$$

is non-expansive that determines γ_1, γ_2 geodesics respectively connecting x to T(x) and y to T(y).

Note: The definition concludes that each firm non-expansive mapping T is non-expansive.

Proposition: Suppose $T: CM \to M$. Then, the following results are equivalent.

1) The T mapping is firmly non-expansive;
2) For every $t \in [1]$, $x, y \in C$

$d(T(x), T(y)) <= d\ (exp_x t exp_x^{-1} Tx,\ exp_y\ te ep_y^{-1} Ty)$

3) For $x, y \in C$

$<exp_{T(x)}^{-1} T(y),\ exp_{T(x)}^{-1} x> + < exp_{T(y)}^{-1} T(x),\ exp_{T(y)}^{-1} y> \leq \bullet$

Proof:

Points $x, y \in C$ are given. Suppose θ: $[0,1] \to [0,1]$ is a convex function. The derivative at -1 of the function θ can be expressed as follows:

$(\theta)\acute{(1)}$ =<$\exp_{T(x)}^{-1}$T(y), $\exp_{T(x)}^{-1}$x> + < $\exp_{T(y)}^{-1}$T(x), $\exp_{T(y)}^{-1}$y>

Suppose u = $\exp_{T(x)}^{-1}$x ∈ $T_{T(x)}$M and v = $\exp_{T(y)}^{-1}$y ∈ $T_{T(y)}$ M, then, the function θ is as follows:

θ (t)=d($\exp_{T(x)}$ (1-t)u, $\exp_{T(y)}$(1-t)v)

Suppose γ is a geodesic that connects T(x) to T (y), as for every $r \in [0,1]$:

$\gamma(r)$ = $\exp_{T(x)}$ r $\exp_{T(x)}^{-1}$T(y)

Now, for the given ϵ > 0, suppose the function $f: (-\epsilon, \epsilon) \times [0,1]$ defined by:

$$f(s,r) = exp_{\exp T\, x^{su}} r\left(exp^{-1}_{expTx^{su}} expT_y\, sv\right). \forall \in (-\epsilon.\epsilon) \times [0,1]$$

Note that for every $s \in (-\epsilon.\epsilon)$, the parametric curve $f_s: [0,1] \to M$ given by $f(s,r) = f_s(r)$ is a geodesic, and therefore, $\frac{\partial f}{\partial r}(s,r)||\ ||$ is constant. In particular, we have:

$||\frac{\partial f}{\partial r}(s,r)||= d\left(expTx^{su}, expT_y\, sv\right) = \theta(1-s)$

We define $l: (-\epsilon.\epsilon) \to R)$

$$l(s) = \int_0^1 ||\frac{\partial f}{\partial r}(s,r)\ ||\, dr, \forall \in s(-\epsilon.\epsilon)$$

Therefore:

$$L^2(s) = \int_0^1 ||\frac{\partial f}{\partial r}(s,r)\ ||\ ^2 dt = \theta^2(1-s)$$

According to the first variable of the equation expressed in [1] where we have $E(s) = 1/2(L^2(s)$

Iterative Algorithm of Non-Expansive Mappings

Picard's Iterations for Firm Non-Expansive Mappings

When Picard's iterations occur in the Banach space and Hilbert sphere with hyperbolic metric [10] [15], as the tier of firm non-expansive mappings will be distinguished by a good symmetrical behavior of Picard's iteration sequence $\{T^n x\}$.

Theorem: Suppose T: C → C is a firm non-expansive mapping whose set of fixed points is Fix(T) ≠ ∅. In this case, for every x ∈ C, there is a sequence of iterations $\{T^n(x)\}$ converging to a fixed point of T.

Proof: A sequence in the form of $x_n = T^n(x)$ is defined; since C is a complete space, therefore, it suffices to denote that $\{x_n\}$ represents the Fejér monotonicity according to Fix (T) and all points of a class of $\{x_n\}$ belonging to Fix (T).

Suppose n=<0 and y∈ Fix(T) is constant. Since T is non-expansive

d(x_{n+1}+,y) = d(T(x_n), T(y)) ≤ d(x_n, y)

then, $\{x_n\}$ is the Fejér monotonicity with Fix (T). Now, suppose x is a point of a class of $\{x_n\}$. So, there is a sub-sequence of $\{n_k\}$ of {n}, with $x_{n_k} \to x$.

So, we need to only prove that:

$$\lim n \to \infty\, d\left(x_n, T(x_n)\right) = 0$$

Because by taking the limit, we have: $\left(x, T(x)\right) = 0$, i.e., x∈ Fix(T)

Suppose y∈ Fix (T). Since $\{x_n\}$ is the Fejér monotonicity concerning Fix(T), there exists the following limit.

$$\lim n \to \infty\, d(x_n, y) = \lim n \to \infty\, d(T(x_n), y) = d$$

Suppose the constant n>=0 is given and $\gamma_n: [0,1] \to M$ is the geodesic that connects x_n to T (x_n). Thus, $\gamma_n(1/2) = m_n$, because T is firmly non-expansive.

$d(T(x_n), y) \leq d(m_n, y) \leq d(x_n, y)$

Hence, lim $d(m_n, y) = d$ are calculated:

$$\frac{1}{4} d^2\big(x_n, T(x_n)\big) \leq \frac{1}{2} d^2(x_n, y) + \frac{1}{2} d^2(x_n, y) - d^2(m_n, y)$$

Taking the limit, when $n \to \infty$, the result is satisfied.

While the T mapping is only non-expansive, it is understood that the Picard iteration $\{T^n(x)\}$ is not usually convergent. By supposing the Euclidean space R and the mapping T (x) =-x where the sequence $\{T^n(x)\}$ is not convergent unless x = ., the sequence defined by the Picard iteration $x_{n+1} = G_t(x_n)$ is taken for every t ∈ [0,1). Moreover, if we hold the point x ∈ C constant, there will be an approximation curve $\{x_t\}$ defined by a single constant point of T_t integration converging into the constant point of T when t → 1.

In fact, this had already been proved by Kirk in a general domain of CAT (0) spaces, forming a convergence extension of Browder's algorithm, which is detailed in the following section.

Halpern Algorithm of Non-Expansive Mappings

Suppose C is a closed convex subset of M and T: C ⟶ C is a non-expansive mapping. To solve the problem of finding a fixed-point T outside of the linear spaces' surface in [16], an implicit algorithm was developed for the approximate fixed points of non-expansive mappings. Although the convergence result of the following theorem is formulated for a specific Hadamard manifold state, an algorithm in the complete CAT (0) space can be studied in detail.

Theorem: (1) Suppose C is a subset of M and is a closed bounded and convex subset; suppose T: C ⟶ C is a non-expansive mapping of x ∈ C, and for every t ∈ [0,1), x_t is a unique point, with

$$x_t = exp_x(1-t)exp_x^{-1}T(x_t)$$

Where this point exists according to the Banach contraction principle; so, $lim_{t\to o} x_t = \bar{x}$ and the closest single point to x is in Fix (T).

In the Euclidean space R^n, the iterative method converts into $x_t = (1-t)x + tT(x_t)$, simultaneously occurring implicitly with the Browder's iteration.

Mann Algorithm of Non-Expansive Mappings

The Mann iteration and a number of the convergence results in Banach spaces to general metric spaces were developed by Goebel and Kirk [17] [18] and Reich and Shafrir [19], who provided an iterative method to find fixed points of non-expansive mappings in the spaces that included Hadamard manifolds as a special state. The algorithm defined with:

$$x_{n+1} \epsilon \big(x_n, T(x_n)\big), d\big(x_n, T(x_n)\big) = (1-\alpha_n) d(x_n, x_{n+1})$$

Where $\big(x_n, T(x_n)\big)$ denotes a metric segment that connects x_n to $T(x_n)$. Strictly speaking, as suggested by the assumption that $\{\alpha_n\}$ is far from 0 and 1 and is bounded, Reich and Shafrir provided this iteration converging to a fixed point of T defined in the Hilbert sphere with the hyperbolic metric. Having said this, the Mann iterations in the Hadamard manifolds M are presented by recursive equations.

$$x_{n+1} = exp_{x_n}(1-\alpha_n)exp^{-1}_{expx_n}T(x_n), \forall n \geq 0$$

Later, it becomes clear that the sequence { produced by the above-mentioned Mann algorithm converges to a fixed point of T, albeit when $\{\alpha_n\}$ satisfies the following condition.

$$\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$$

Numerical Example

Suppose E ^{m,1} is the vector space R $^{m+1}$. Let's consider the following symmetrical bilinear form:

$$< x, y >= \sum_{i=1}^{m} x_i y_i - x_{m+1}y_{m+1}, \forall x = (x_i), y = (y_i) \in R^{m+1}$$

This bilinear form is the Lorentzian metric. M is the hyperbolic space H^m defined by:

$$\{x = (x_i, \dots, x_{m+1}) \in R^{m+1}: < x, x \geq -1, x_{m+1} > 0\}$$

Note that $1 \leq x_{m+1}$ is for every x∈H^m, and is equal if and only if for every m, i= $-$ 1000 and $x_i = 0$ of the H^m metric are inferred from the Lorentzian metric <0,0>. H^m is a Hadamard manifold with local curvature -1 [20] and [18]. In addition, the geodesic normalized by $\gamma: R \to H^m$ starting from x∈H^m is given by

$$\gamma(t) = (\cosh t)x + (\sinh t)v \ \ , \forall t \epsilon R$$

Where $v \in T_x H^m$ is a single vector, and the distance d is defined on H^m. The exponential mapping can thus be expressed as follows:

$$exp_x(rv) = (\cosh r)x + (\sinh r)v$$

For every $r \in R^+$ and x∈H^m, and every single vector $v \in T_x H^m$, we write the following for every x, y ∈H^m:

$$exp_y^{-1} = (-< x, y >)\frac{y+< x, y > x}{\sqrt{< x, y >^2 - 1}}, , \forall x, y \epsilon H^m$$

Therefore, the Halpern algorithm takes on the form below:

$$x_{n+1} = \cosh\big((1-\alpha_n)r(u, x_n)\big)u + \sinh\big((1-\alpha_n)r(u, x_n)\big)V(u, x_n) \ \ , \forall n \geq 0$$

The Mann algorithm will truly take on the form below:

$$x_{n+1} = \cosh\big((1-\alpha_n)r(u, x_n)\big)u + \sinh\big((1-\alpha_n)r(x_n, x_n)\big)V(x_n, x_n) \ \ , \forall n \geq 0$$

Saddle Points on the Minimal Problem

Rockefeller's work [21] on the convergence of the origin point algorithm is when an associative maximal monotone operator acts for the saddle points in the Hilbert space product $H_1 \times H_2$. The section focuses on the convergence of the origin point algorithm of saddle functions on Hadamard manifolds.

Suppose M$_1$ and M$_2$ are Hadamard manifolds. The function $L: M_1 \times M_2 \to R$ is called saddle, if, for every $L(x, 0), x \in M_1$, it is convex on M2 and concave on $L(0, y)$, denoting that for every y ∈M_2, $-L(0, y)$ is convex on M$_1$. The point $\bar{z} = (\bar{x}, \bar{y}) \in M_1 \times M_2$ is called the saddle point L, when:

$$L(x, \bar{y}) \leq L(\bar{x}, \bar{y}) \leq L(\bar{x}, y) \quad , \forall z = (x, y) \in M_1 \times M_2$$

The associative property of the saddle functions L defines the value set $A_L: M_1 \times M_2 \to 2^{TM_1} \times 2^{TM_2}$ with

$$A_L(x,y) = \delta\big(-L(0,y)\big)(x) \times \delta\big(L(x,0)\big)(y), \qquad \forall(x,y) \in M_1 \times M_2$$

As cited by the [12] p. 239 of Problem 10], the product space MM is the Hadamard manifold and the tangent space of M is in $T_zM = T_xM_1 \times T_yM_2 . z(x,y)$, the corresponding metric is expressed as follows:

$$< \omega, \acute{\omega} >=< u, \acute{u} > + < v, \acute{v} > , \forall \omega = (u,v), \acute{\omega} = (\acute{u}, \acute{v}) \in T_zM$$

Note that a geodesic in the product set M is the product of two geodesics in M_1 and M_2, respectively. So, for every two points $z(x,y)$ and $\acute{z} = (\acute{x}, \acute{y})$ in M:

$$exp_Z^{-1}\acute{z} = (exp_Z^{-1}\acute{x}, exp_Z^{-1}\acute{y})$$

Therefore, defining the vector field monotonicity, the value set $A : M_1 \times M_2 \to 2^{TM_1} \times 2^{TM_2}$ is monotone, and only if:

$$\acute{\omega} = (\acute{u}, \acute{v}) \in A(\acute{z}), \omega = (u,v) \in A(z), \qquad z(x,y) \text{ و} \acute{z} = (\acute{x}, \acute{y})$$

$$< u, exp_x^{-1}\acute{x} > + < v, exp_y^{-1}\acute{y} > \leq < \acute{u}, exp_x^{-1}\acute{x} > + < \acute{v}, exp_y^{-1}\acute{y} >$$

Variational Inequality

Suppose C is a convex subset of M and $V: C \to TM$ is a unit vector field where for every C $V(x) \in T_zM$, as based on [22], the problem of finding $x \in C$ with the condition

$$< V(x), exp_x^{-1}y >= 0. \ \forall(y) \in C$$

Is referred to as a variational inequality in C. Clearly, the point $x \in C$ is a solution of the above-mentioned variational inequality, if and only if x is in a way that

• ∈ V(x) + N_C(x)

, denoting x is the singularity of the vector field of the value set A = V + N_C. Using the Mann algorithm for A, we have the lower origin point algorithm with the main point to find the variational inequality solution $< V(x), exp_x^{-1}y \geq 0. \ \forall(y) \in C$

$$0 \in V(x_{n+1}) + N_C(x_{n+1}) - \lambda_n exp_x^{-1} \text{ و} \forall n \geq 0$$

Proposition: Suppose C is the convex compact subset of M, so, there exists a general geodesic sub-manifold N as the subset of C, with $\bar{N} = C$ (closure) and the following condition is met for every $q \in C/N$ and p∈ N; for every $(0.1) exp_p\, t(exp_p^{-1}q) \in N$ and for every $exp_p\, t(exp_p^{-1}q) \notin C, \ t \in (1, \infty)$.

Note: According to [12], int C:N is the interior points of C and bdC: =C \N is the boundary points of C. In addition, if C is the convex compact set, then, bdC≠0.

References

[1] S. Z. N´emeth, Variational inequalities on Hadamard manifolds, Nonlinear Anal. 52 (2003), 1491-1498.

[2] J. M. Borwein and A. S. Lewis, Convex Analysis and Nonlinear Optimization, Theory and Examples, Springer, 2006.

[6] Y. Censor, T. Elfving, N. Kopf and T. Bortfeld, The multiple-sets split feasi- bility problem and its applications for inverse problems, Inverse Problems 21 (2005), 2071-2084.

[7] P. L. Combettes and S. A. Hirstoaga, Equilibrium programming in Hilbert Spaces, Journal of Nonlinear and Convex Analysis 6 (2005), no. 1, 117-136.

[9] J. X. Da Cruz Neto, O.P. Ferreira and L.R. Lucambio P' erez, Contributions to the study of monotone vector fields, Balkan Journal of Geometry and Its Applications 3 (2002), no. 1, 207-218.

[10] O.P. Ferreira and P.R. Oliveira, Proximal point algorithm on Riemannian manifolds, Optimization 51 (2002), no. 2, 257-270.

[14] T. Inamiya and H. Okochi, Monotonicity, resolvents and Yosida approximation on Hilbert manifolds, Nonlinear Anal. 54 (2003), 205-214.

[17] W.A. Kirk, Geodesic geometry and fixed-point theory, II. International Conference on Fixed Point Theory and Applications, 113-142, Yokohama Publ., Yokohama, 2004.

[18] C. Li, G. López and V. Martín-Márquez, Proxim´e e et dualit´e dans un espace Hilbertien, Journal of London Mathematical Society 79 (2009), no. 2, 663-683.

[27] T. Rapcsák, Sectional curvature in nonlinear optimization, J. Global Optim. 40 (2008), no. 1-3, 375-388.

[13] K. Goebel and S. Reich, Uniform Convexity, Hyperbolic Geometry, and Non- expansive Mappings, Marcel Dekker, New York, 1984.

[28] S. Reich and I. Shafrir, Non-expansive iterations in hyperbolic spaces, Nonlinear Anal. 15 (1990), no. 6, 537-558.

[33] T. Sakai, Riemannian Geometry, Translation of Mathematical Monographs 149, American Mathematical Society, Providence, RI, 1996.

[4] R. E. Bruck, Convergence theorems for sequence of nonlinear operators in Banach spaces, Math. Z. 100 (1967), 201-225.

[5] R. E. Bruck, Non-expansive projections on subsets of Banach spaces, Pac. J. Math 47 (1973), 341-355.

[29] S. Reich and I. Shafrir, The asymptotic behavior of firmly non-expansive mappings, Proc. Amer. Math. Soc. 101 (1987), no. 2, 246-250.

[15] W. A. Kirk, Geodesic Geometry and Fixed-Point Theory, Seminar of Mathematics (Malaga/Sevilla, 2002/2003), 195-225, Univ. Sevilla Secr. Publ., Sevilla, 2003.

[16] W. A. Kirk, Krasnoselskii's Iteration process in hyperbolic space, Numerical Functional Analysis and Optimization 4 (1981/82), no. 4, 371-381.

[12] K. Goebel and W. A. Kirk, Iteration processes for non-expansive mappings, Contemporary Mathematics 21 (1983), 115-123.

[8] J. X. Da Cruz Neto, O.P. Ferreira and L.R. Lucambio P' erez, Monotone point-wise-Lipschitz mappings in Banach spaces, Journal of Global Optimization 5 (2000), 69-75.

[11] O.P. Ferreira, L.R. Lucambio P' erez and S.Z. N' emeth, Singularities of monotone vector fields and an extragradient-type algorithm, J. Global Optim. 31 (2005), 133-151.

[31] R. T. Rockafellar, Monotone operators and the proximal point algorithm, SIAM J. Control Optim. 14 (1976), no. 5, 877-898.