

Analysis of pressure wave dynamics in fuel rail system

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ABSTRACT

A model of an amplified common rail fuel system is simulated in Matlab to analyze the wave mechanics in the rail. The injectors are modeled as a system of linear and non-linear ODE's consisting of masses, a helical spring, compressibility effects from fluid volumes, and hydraulic flow through orifices. The injector simulation then predicts the rate of oil consumption, which is then input into the rail model.

The rail is modeled in three sections which are coupled together. The points where the coupling occurs are the locations where the current firing injector and the pump supply are connected to the rail. This allows the model to control the pressure and velocity (as boundary conditions) at these points. The rail model is based on the 1D, undamped wave equation, in a non-dimensional form [1] (in the position variable, x .) The Reduction of Order method was used to solve the wave equation with the Matlab function PDEPE.

The model was run with two different sets of initial conditions - nominal (constant pressure and zero velocity,) and worst case using a simplified representation of the pressure and velocity distribution at start of injection. This was done to determine the effect of rail waves at the start of injection, on the output of the model. The variation in fuel delivery, due to the variation in rail pressure, was then evaluated at three operating conditions - Idle, Peak Torque (PT) and High Speed Light Load (HSL.) The simulation output is then compared to analytical solutions of two forms of simplified geometry, using the product method to solve the system [1.]

1. INTRODUCTION

Modern Diesel fuel systems utilize a common rail to distribute pressurized fuel to unit injectors. Previous generation technology commonly used unit pumps, which were driven from camshafts. This had the disadvantage of limiting injection pressure control (because the injection pressure is approximately proportional to engine speed.) Common Rail fuel systems operate independent of engine speed. This adds greater control for engine management and emissions reduction.

A hybrid approach to the common rail injector is the Amplified Common Rail (ACR.) Designs utilizing a hydraulic amplification system to boost the injection pressure (secondary) typically use the energy from a supply pressure (primary.) The advantage with an ACR design is that higher injection pressure (>200 Mpa) can be achieved. The higher injection pressures produce a better atomization of the fuel which then reduces the formation of NO_x and particulates during combustion.

Variation in the rail pressure however increases the variation in fuel delivery [2, 3] and increases emissions output from the engine.

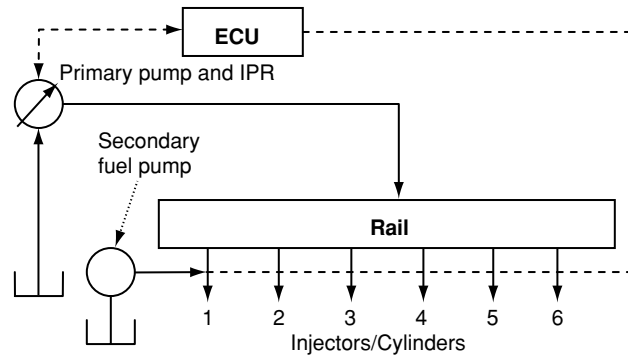


Figure 1 Schematic of an ACR system.

Analyzing the system behavior at Idle, PT and HSLL is useful because these represent common conditions in vehicle and engine load histories.

2. SYSTEM MODEL

A schematic of an ACR system is shown in Figure 1. A primary system pump supplies pressurized oil to the rail (up to 4000 psi.) A secondary system pump supplies low pressure (60 psi) fuel to the injectors. The injection pressure regulator (IPR,) and the injector control valves are controlled by the Engine Control Unit (ECU.) The primary supply pump is a variable displacement pump.

3. INJECTOR MODEL

A section view of an ACR injector is shown Figure 2 with the control valve spool is shown in a neutral position. When the ECU supplies current to the open coil, the magnetic force pulls the spool to the open position (left.) Fluid then flows from the supply port, through the valve and into the intensifier. The supply pressure then produces a force on the top surface of the piston which drives the piston downward. The force transmits through to the plunger which then compresses the diesel fuel in the lower section of the intensifier. The nozzle spring exerts a downward force on the needle. The pressure necessary to overcome the nozzle spring force, and lift the needle, is the Valve Opening Pressure (VOP.) As the fuel pressure in the lower end of the intensifier increases above VOP, the needle lifts off of the seat in the nozzle, and fuel flows through the orifices in the nozzle and into the engine cylinder.

At the end of injection, the ECU supplies current to the close coil. This causes the spool to shift to the close position (right.) The intensifier spring then pushes the piston and plunger upward, which causes the primary fluid to vent from the space above the piston. This also causes the inlet check ball to lift off of its seat and fuel to flow into the lower intensifier bore. After the piston has reached the top of the bore (in contact with the control valve body) the injector is ready for the next injection.

A schematic representation of the injector and rail is shown in Figure 3. The pressure in the rail can be modeled from the compressibility equation, where P_r is the rail pressure, B is the Bulk Modulus, V_r is the rail volume, Q_p is the pump flow, Q_o is the injector flow, and t is time.

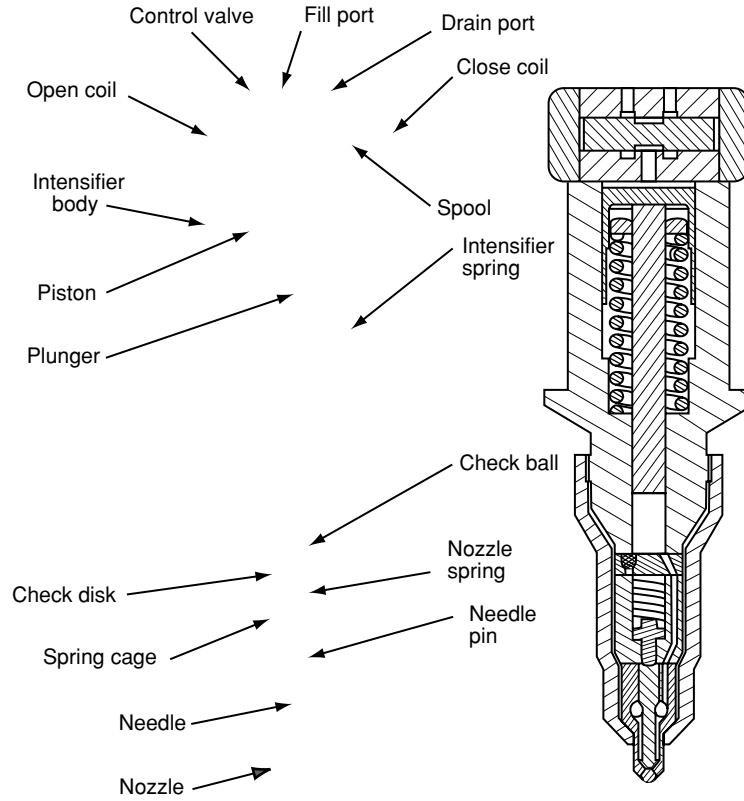


Figure 2 Section view of an ACR injector.

$$dP_r = \frac{B}{V_r} (Q_p - Q_o) dt \quad (1)$$

The flow through the control valve can be modeled as orifice flow,

$$Q_o = A_c \sqrt{\frac{2(P_r - P_o)}{\rho}} \quad (2)$$

where A_c is the flow area across the spool, P_o is the pressure above the piston and ρ is the density of the fluid.

The pressure above the piston can be modeled as

$$dP_o = B \frac{Q_o dt - A_o dx}{V_o + A_o x} \quad (3)$$

where A_o is the area of the piston, x is the piston motion, and V_o is the residual volume between the piston and the spool (when the piston is at the top of its bore,)

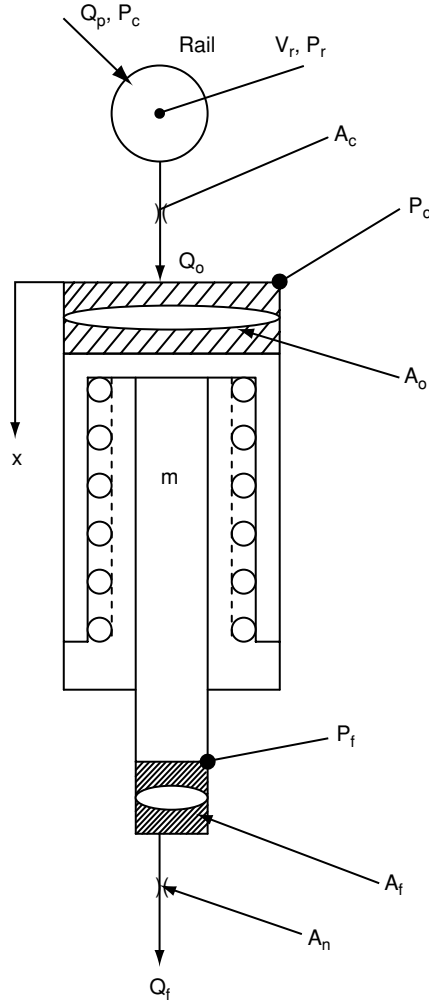


Figure 3 Schematic of an ACR injector.

The motion of the piston/plunger system can be modeled as a mass-spring system,

$$m\ddot{x} + Kx = P_o A_o - P_f A_f - F_o \quad (4)$$

where m is the combined mass of the piston and plunger, K is the intensifier spring rate, P_f is the fuel pressure and A_f is the plunger area.

The fuel pressure can be modeled as

$$P_f = B \frac{A_f dx - Q_f dt}{V_f - A_f x} \quad (5)$$

where Q_f is the fuel flow and V_f is the volume of fuel in the lower section at the start of injection.

The fuel flow can be modeled as

$$Q_f = A_n \sqrt{\frac{2P_f}{\rho}} \quad (6)$$

where A_n is the nozzle orifice flow area.

The injector model is then solved with the system of equations 1-6 using Matlab.

3.1. RAIL MODEL

The wave mechanics in the rail can be modeled with the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

The Matlab function PDEPE solves elliptic and parabolic partial differential equations (PDE's). The wave equation however is a hyperbolic PDE. PDEPE can be used to solve the wave equation by applying reduction of order and then solving a system of PDE's. Substituting

$$v = \frac{\partial u}{\partial t} \quad (7)$$

into the wave equation gives

$$\frac{\partial v}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad (8)$$

The pressure in the fluid is

$$P = -B \frac{\partial u}{\partial x} \quad (9)$$

The minus sign in Eq 9 indicates that “tension” is negative pressure and “compression” is positive. Differentiating Eq 7 with respect to x gives

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial t} \right) = \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial x} \right) = \frac{\partial}{\partial t} \left(\frac{P}{-B} \right).$$

Multiplying both sides by -B gives

$$\frac{\partial P}{\partial t} = -B \frac{\partial v}{\partial x} \quad (10)$$

Multiplying both sides of Eq 8 by -B and substituting Eq 9 into Eq 8 gives

$$-B \frac{\partial v}{\partial t} = c^2 \frac{\partial P}{\partial x} \quad (11)$$

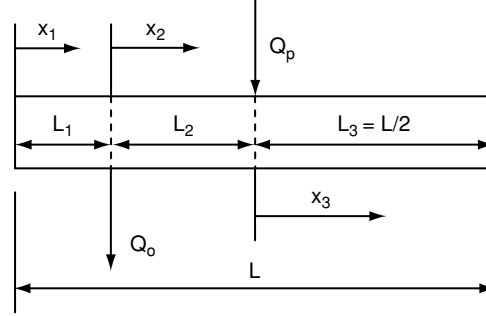


Figure 4 Schematic of rail geometry.

The system of equations 10 and 11 can then be solved given initial and boundary conditions for pressure and velocity. A schematic representation of the rail is shown in Figure 4. The injectors are connected at equally spaced positions along the rail corresponding to the engine cylinder locations in the block and head. Only one injector position is shown because only one injector delivers fuel at a given point in time.

The rail is divided into three sections, each of a different length. The PDEPE module in Matlab can solve a system of equations for all three sections, simultaneously. This can be done by making the equations non-dimensional in the x -coordinate, by dividing each x in equations 10 and 11 by their respective lengths, where the non-dimensional coordinate,

$$X = \frac{x_1}{L_1} = \frac{x_2}{L_2} = \frac{x_3}{L_3} \quad [1, \text{pg } 1102]$$

The coordinate X then varies between 0 and 1 for all three systems. The system of equations is then:

$$\begin{aligned} -\frac{L_1 B}{c^2} \frac{\partial v_1}{\partial t} &= \frac{\partial P_1}{\partial X} \text{ and } -\frac{L_1}{B} \frac{\partial P_1}{\partial t} = \frac{\partial v_1}{\partial X} \\ -\frac{L_2 B}{c^2} \frac{\partial v_2}{\partial t} &= \frac{\partial P_2}{\partial X} \text{ and } -\frac{L_2}{B} \frac{\partial P_2}{\partial t} = \frac{\partial v_2}{\partial X} \\ -\frac{L_3 B}{c^2} \frac{\partial v_3}{\partial t} &= \frac{\partial P_3}{\partial X} \text{ and } -\frac{L_3}{B} \frac{\partial P_3}{\partial t} = \frac{\partial v_3}{\partial X} \end{aligned} \quad (12)$$

The system needs to be coupled at the two common boundaries, $x_2 = 0$ and $x_3 = 0$. The pressure is equal on both sides of these boundaries so that, in non-dimensional form, $P_{1(1,t)} = P_{2(0,t)}$ and $P_{2(1,t)} = P_{3(0,t)}$.

The flow across these boundaries must be equivalent to the flow from the pump and the flow to the injector. The flow can be divided by the cross-sectional area of the rail to give the velocity. The velocity boundary conditions are then $v_o = v_{1(1,t)} - v_{2(0,t)}$ and $v_p = v_{3(0,t)} - v_{2(1,t)}$, as shown below in Figure 5.

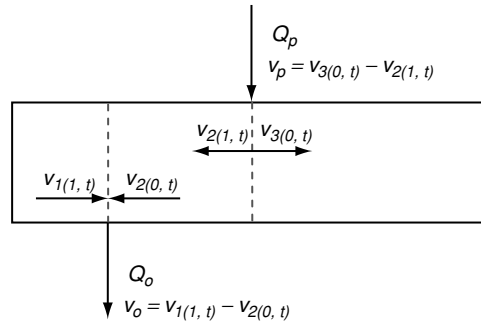


Figure 5 Velocity boundary conditions.

The remaining two boundary conditions are zero velocity at $x_1 = 0$ and $x_3 = L/2$. This is equivalent to fixing the end points of the rail in a fixed-fixed system, and can be modeled in PDEPE as $v_{1(0,t)} = 0$ and $v_{3(1,t)} = 0$. The rail model is then solved in the Matlab. The physical geometry of the rail is a 1 meter long tube with a port in the middle (at 500 mm) for the pump supply, two ports 125 mm from each end (#1 and #6 injectors,) and the remaining injectors (#2 - #5) 150 mm apart. The rail volume is 0.7 L and the cross-sectional area is 7 cm².

3.2. RESULTS

The results from running the injector simulation under Peak Torque operating Conditions are shown in Figure 6.

The upper pane in Figure 6 is the Rail Pressure, the middle pane is the injection pressure, and the lower pane is the Rate of Injection (ROI). The rate of injection is simply the instantaneous flow rate of fuel into the engine.

Two of the simplifications that were used in the injector model are 1. Instantaneous motion of the control valve spool, and 2. Needle (nozzle valve) always opens. Modeling the motion of the spool requires analysis of the magnetic and the dynamics, and takes the form of an “on-off” non-linearity. Modeling the needle motion also requires additional dynamic analysis and is also non-linear. Both of these effects were omitted in order to simplify the model.

Fuel injectors typically also have an additional check valve between the pumping unit and the needle – the Reverse Flow Check (RFC.) The purpose of the RFC is to prevent reverse flow of exhaust gases, into the injector. This can occur at the End of Injection (EOI,) after the control valve is closed and while the needle is still open. The motion of the RFC also is non-linear.

A simplified representation of the injector ROI is shown in Figure 7. The EOI is typically an exact reversal of the start of injection (SOI.) The SOI and EOI are modeled in the rail simulation as

$$Q_{soi} = Q_{ss} \left(1 - e^{-t/t_c} \right) \text{ and } Q_{eoi} = Q_{ss} e^{(t-t_d)/t_c} \quad (13)$$

where t_c is the time constant and t_d is the “on time” (time duration of the injection.) Equations 13 are used to turn on the injection in the Boundary Conditions. The oscillations that occur at the beginning of the ROI in figure 6 are typical of real systems. These oscillations also typically occur in the EOI. The oscillations could be produced in the model by adding a decayed sine function to the equation. This effect, and a model of the RFC were also omitted in order to simplify the model.

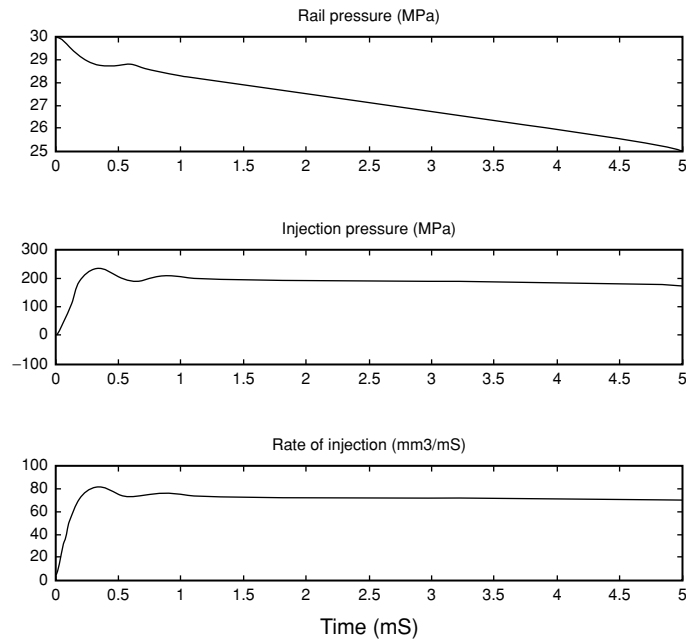


Figure 6 Injector.m output at Peak Torque.

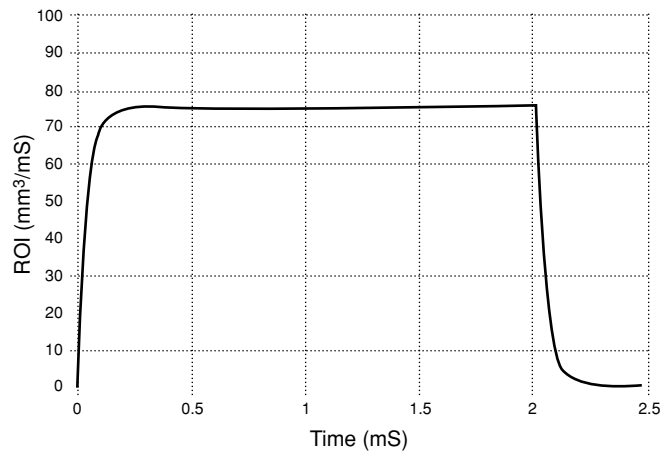


Figure 7 Simplified representation of ROI.

In Matlab M-file one can calculate the pressure and velocity distribution in the rail from a single injection event. The output is in the form of surface maps (pressure as a function of time and rail position) and x-y plots of pressure as a function of time. The surface plots give an overall view of the wave propagation in the rail. A surface plot of the response from Injector 1 firing under Peak Torque operation is shown in Figure 8. The response is a result of “bumps” in the surface propagating outward from the pump location, and “dips” in the surface propagating outward from the injector location, at the wave speed

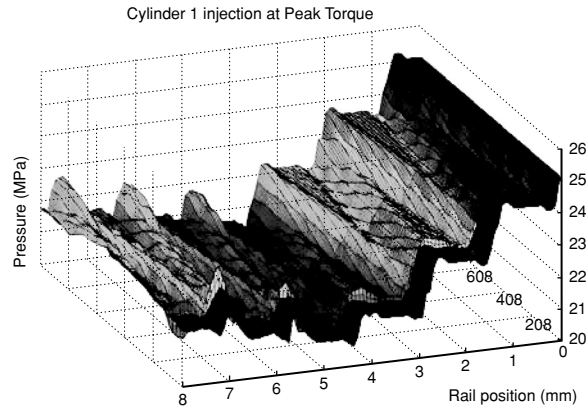


Figure 8 Rail response due to Injector 1 at Peak Torque.

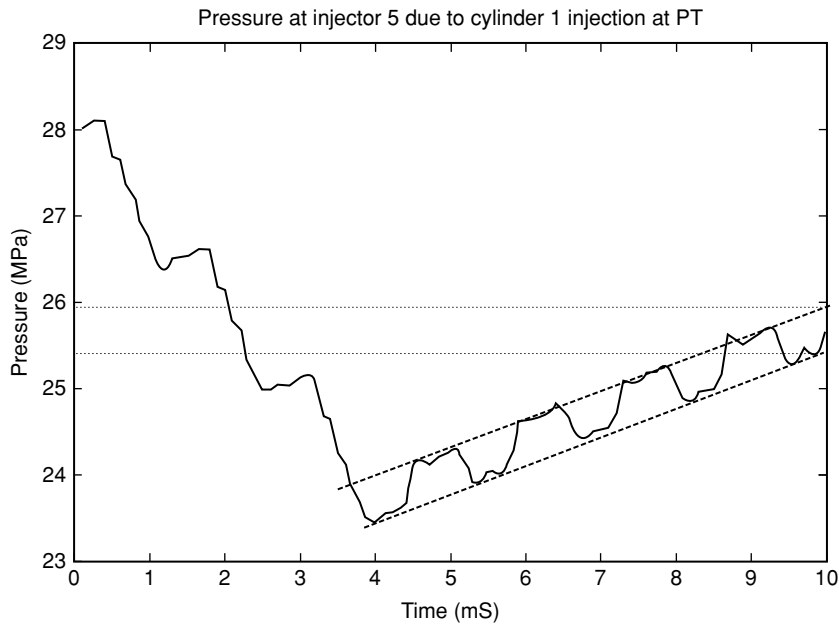


Figure 9 Effect of cylinder 1 on cylinder 5 at PT.

$$c = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{1.7 \text{ GPa}}{0.82 \text{ gm/cc}}} = 1440 \text{ m/s}$$

The objective of this study is to evaluate the effect of the response on successive injectors firing order is 1-5-3-6-2-4, respectively. Therefore, we need to examine the response due to cylinder 1 at the location of cylinder 5. This is shown in Figure 9. The simulation was run

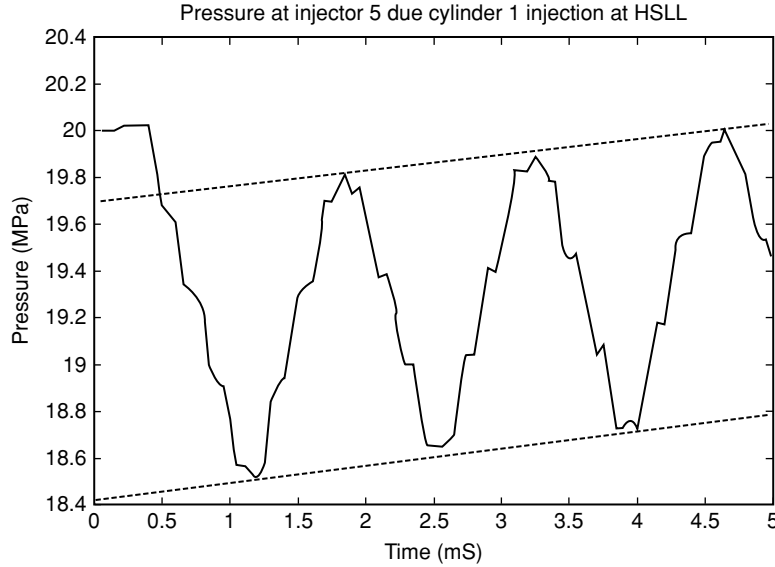


Figure 10 Effect of cylinder 1 on cylinder 5 at HSL.

for 4 mS of injection, and an additional 6 mS after EOI. During the injection event, the pressure decreases because the injector flow rate (out flow) is greater than the pump flow rate (in flow.) After EOI, the pressure recovers because the flow to the injector has stopped. If the simulation were continued for another ~6 mS, the pressure would fully recover to the control pressure (in this case 28 MPa,) after which the pump control would hold the pressure constant (with a certain amount of steady state error.)

The engine speed at PT is 1200 rpm. The time between injections is therefore

$$t = \frac{1 \text{ min}}{1200 \text{ revs}} \left(\frac{1 \text{ rev}}{3 \text{ inj}} \right) \left(\frac{60 \text{ S}}{\text{min}} \right) \left(\frac{1000 \text{ mS}}{\text{S}} \right) = 16.7 \text{ mS/inj}$$

The exact timing of the next injection however is dependant on the exact operating condition. The time it takes to move from a valley to a peak in Figure 9 is approximately 0.3 mS. The timing at 1222 rpm is 16.4 mS per injection. So a difference in engine speed of 22 rpm would make the difference between starting the next injection at 28.3 vs 27.7 MPa. Therefore, the amplitude of the wave in the recover portion of Figure 9 is the measure of variability on the system. The responses due to Cylinder 1 at HSL and Engine Idle are in Figures 10 and 11.

Similar data can be plotted for the remaining cylinders. Table 1 is a summary of the analysis conditions, where P_c is the control pressure (MPa,) v_f is the fuel delivery (mm^3), ROI is the rate of injection (mm^3/mS), speed is the engine speed (rpm,) v_o is the velocity of oil to the injector (m/S,) v_p is the velocity of oil from the pump (m/S) and t_d is the injection duration (mS.)

Table 2 is a summary of the results. Only data for cylinders 1–3 are plotted because, by symmetry, cylinders 4–6 would produce the same result.

PDEPE requires one initial condition for each of the six PDE's in the system. For the velocities, the initial velocity was set to zero, and the pressure was set to P_c . In a real system however the initial state would depend on the final state of the previous injection cycle. The

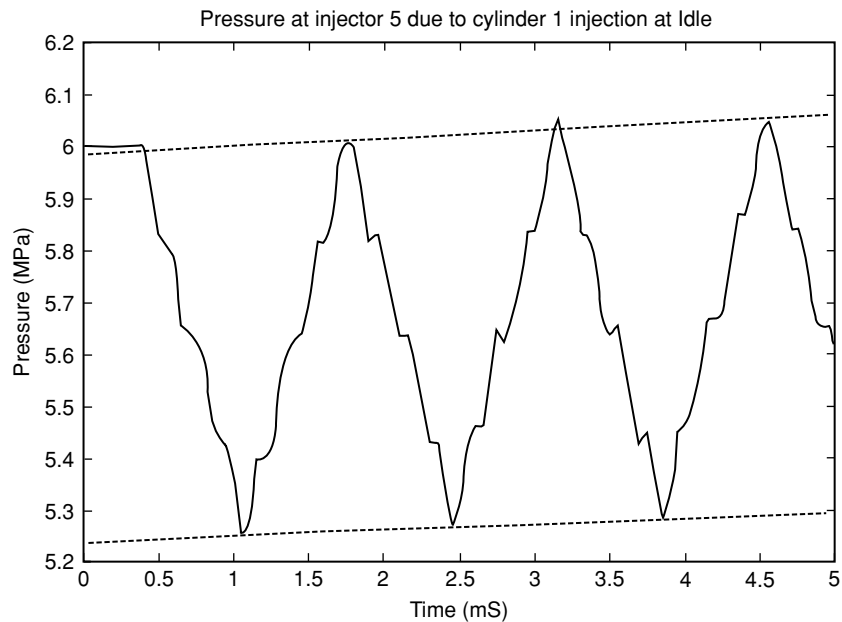


Figure 11 Effect of cylinder 1 on cylinder 5 at Idle.

Table 1 Analysis conditions

State	P_c	v_f	ROI	Speed	v_o	v_p	t_d
Idle	6	20	30	600	0.33	0.007	0.67
PT	28	300	75	1200	0.82	0.196	4
HSL	20	50	60	1500	0.65	0.041	0.83

Table 2 Variation in pressure during pumping recover

State	Cylinder 1	Cylinder 2	Cylinder 3
Idle	0.75	0.35	0.35
HSL	1.3	0.4	0.6
PT	0.6	0.6	0.6

only way to input a pressure and velocity distribution into PDEPE is with analytical functions (functions of x and t .) This is not practical to do for several cycles. However it can be done for one injection, to determine the effect.

A graph of the pressure (solid curve) along the length of the rail after cylinder 4 fired is shown in Figure 12. The dashed curve is an analytical expression that approximates the pressure. Figure 13 is a similar set of curves representing the velocity distributions. These functions are calculated in the file inputs.xls.

Inputting these expressions into the initial condition in PDEPE produced the results shown in Figure 14 and 15.

Figure 14 is a surface plot of the pressure vs. rail position and time. Figure 15 is a plot of the pressure drop during injection, and the recover after EOI.

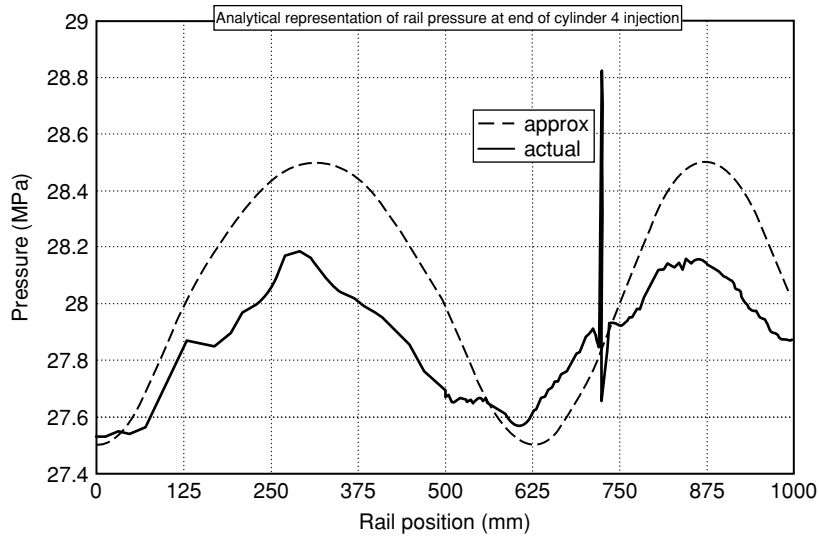


Figure 12 Pressure input to injector 1.

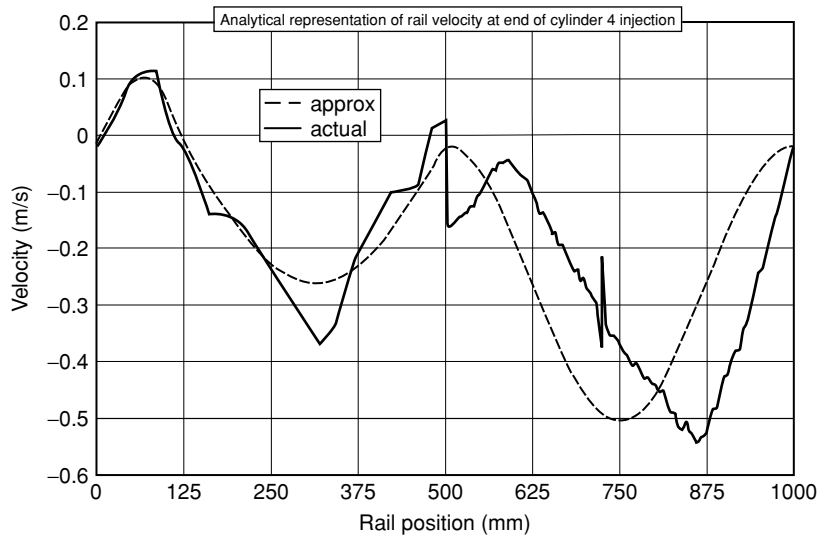


Figure 13 Velocity input to injector 1.

Both plots show a clear influence from the previous cycle with additional wave effects evident in the surface plot and increased variability in the P vs. t plot (from ± 0.35 MPa to ± 0.5 MPa.)

The variation in fuel delivered by the injector, due to the variation in pressure, can be determined from the injector model by integrating the ROI trace over the injection duration. Inputting the min and max projected pressure at the end of the recovery into the injector model produced the results summarized in Table 3.

The largest variation in fuel delivery occurred at PT with ± 1.5 mm³. HSLI and Idle were ± 0.75 and ± 0.75 mm³. The HSLI and Idle points should be acceptable for engine operation.

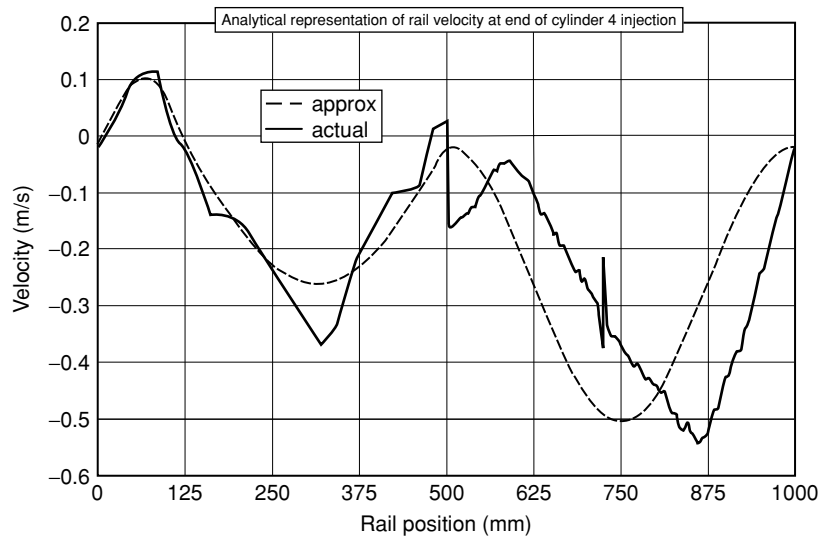


Figure 14 Pressure waves due to injector 1 (with IC's).

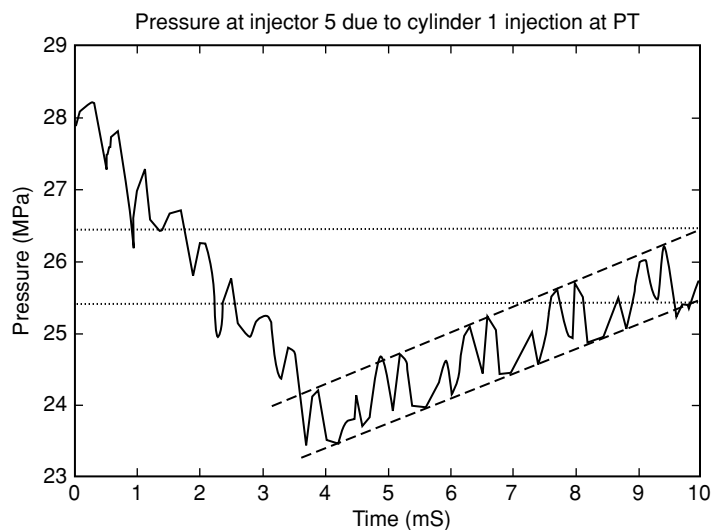


Figure 15 Effect of cylinder 1 on cylinder 5 (with IC's).

The PT variability does typically operate that high, but needs to be lowered in order to achieve future emissions standards.

3.3. COMPARISON TO ANALYTICAL SOLUTION

The model of the rail is a fixed-fixed system with velocity constraints at the injector and pump connections. An exact analytical model of the rail, with injector and pump inputs, would be difficult to produce because it requires simultaneous solution of a system of PDE's. However, it is possible to compare the numeric solution for two simplified models of the rail: a fixed-fixed and a fixed-free model.

Table 3 Variation in fuel delivery due to pressure

Peak Torque		HSL		Idle	
P (MPa)	v(mm ³)	P (MPa)	v(mm ³)	P (MPa)	v(mm ³)
27.7	272.4	19.35	44.7	5.6	18.3
28	273.9	20	45.5	6	19.1
28.3	275.5	20.65	46.2	6.4	19.8

It can be shown that the solution to a fixed-free system is

$$P = -4P_o \sum_{n=1}^{\infty} \cos n\pi \cos \frac{(2n-1)\pi x}{2L} \cos \omega t \quad [4, \text{pg 433}]$$

The solution method is presented here in a slightly different form than [4.] A general solution of the wave equation can be expressed as:

$$u_{(x,t)} = \left(A \sin \frac{\omega}{c} x + B \cos \frac{\omega}{c} x \right) (C \sin \omega t + D \cos \omega t)$$

With one end of the rail fixed, and the other end free, the boundary conditions are

$$u_{(0,t)} = 0 \text{ and } \frac{\partial u}{\partial x}_{(L,t)} = 0$$

Substituting the first condition into the general solution gives

$$u_{(0,t)} = B(C \sin \omega t + D \cos \omega t) = 0 \therefore B = 0$$

Substituting the second condition into the general solution gives

$$\frac{\partial u}{\partial x}_{(L,t)} = \left(A \frac{\omega}{c} \cos \frac{\omega}{c} L \right) (C \sin \omega t + D \cos \omega t) \therefore \cos \frac{\omega}{c} L = 0 \therefore \frac{\omega}{c} L = \frac{k\pi}{2}, k = 1, 3, 5 \dots \therefore$$

The natural frequencies are then

$$\omega = \frac{(2n-1)\pi c}{2L}, n = 1, 2, 3 \dots$$

The general solution can then be rewritten in the form

$$u_{(x,t)} = \left(\frac{v_o}{\omega} \sin \omega t + u_o \cos \omega t \right) \sin \frac{\omega}{c} x$$

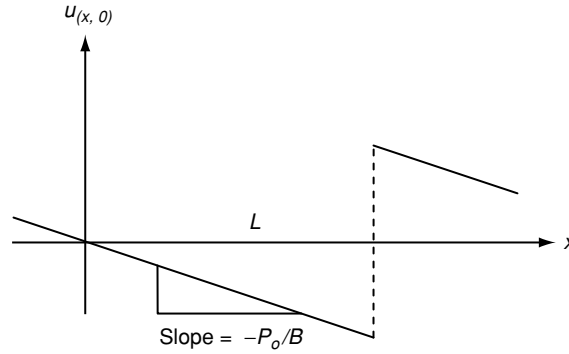


Figure 16 Saw tooth wave initial condition.

where u_o and v_o are the displacement and velocity at time zero. The velocity profile is then

$$v_{(x,t)} = \dot{u}_{(x,t)} = (v_o \cos \omega t - u_o \omega \sin \omega t) \sin \frac{\omega}{c} x$$

The initial conditions are

$$v_{(x,0)} = 0 \text{ and } u_{(x,0)} = -\frac{P_o}{B} x$$

The velocity at time zero is

$$\dot{u}_{(x,0)} = v_o \sin \frac{\omega}{c} x = 0 \therefore v_o = 0$$

The general solution can then be rewritten in the form

$$u_{(x,t)} = u_o \sin \frac{\omega}{c} x \cos \omega t$$

For the saw tooth wave shown in Figure 16, the initial condition

$$u_{(x,0)} = -\frac{P_o}{B} x$$

Can be expanded through Fourier series representation as

$$-\frac{P_o}{B} x = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \frac{n\pi x}{L}$$

where

$$b_n = \frac{2}{L} \int_0^L -\frac{P_o}{B} x \sin \frac{n\pi x}{L} dx = -\frac{P_o}{B} \frac{2}{L} \int_0^L x \sin \frac{n\pi x}{L} dx$$

From integration by parts,

$$\begin{aligned} \int x \sin \frac{n\pi x}{L} dx &= -x \frac{L}{n\pi} \cos \frac{n\pi x}{L} - \int -\frac{L}{n\pi} \cos \frac{n\pi x}{L} dx \\ &= \left(\frac{L}{n\pi} \right)^2 \sin \frac{n\pi x}{L} - x \frac{L}{n\pi} \cos \frac{n\pi x}{L} \end{aligned}$$

One can find,

$$\begin{aligned} b_n &= -\frac{P_o}{B} \frac{2}{L} \left(\frac{L}{n\pi} \right)^2 \sin \frac{n\pi x}{L} - x \frac{L}{n\pi} \cos \frac{n\pi x}{L} \Bigg|_0^L \\ &= -\frac{P_o}{B} \frac{2}{L} \left[0 - \frac{L^2}{n\pi} \cos n\pi \right] = 2 \frac{P_o}{B} \frac{L}{n\pi} \cos n\pi \end{aligned}$$

The initial condition can then be represented as

$$-\frac{P_o}{B} x = 2 \frac{P_o}{B} \frac{L}{\pi} \sum_{n=1}^{\infty} \frac{\cos n\pi}{n} \sin \frac{n\pi x}{L}$$

The overall solution is the sum of all possible solutions which can be expressed as

$$u_{(x,t)} = \sum_{n=1}^{\infty} u_n \sin \left(\frac{(2n-1)\pi x}{2L} \right) \cos \omega t$$

The initial condition is then

$$u_{(x,0)} = \sum_{n=1}^{\infty} u_n \sin \left(\frac{(2n-1)\pi x}{2L} \right) = 2 \frac{P_o}{B} \frac{L}{\pi} \sum_{n=1}^{\infty} \frac{\cos n\pi}{n} \sin \frac{n\pi x}{L}$$

These two series do not have the same form and therefore the u_n constant cannot be evaluated.

If however, instead of using the saw tooth wave of Figure 16, we use the triangle wave of Figure 17, we obtain the following:

$$u_{(x,0)} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2L}$$

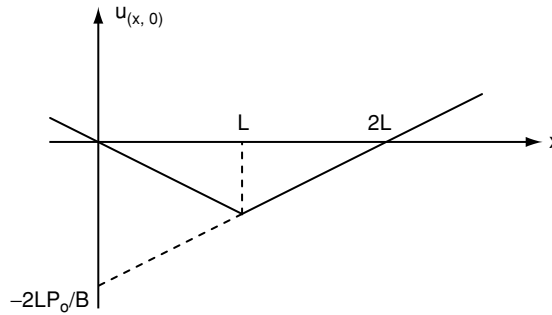


Figure 17 Triangle wave initial condition.

where the Fourier coefficient is:

$$b_n = \frac{2}{2L} \int_0^L -\frac{P_o}{B} x \sin \frac{n\pi x}{2L} dx + \frac{2}{2L} \int_L^{2L} \left(-2\frac{P_o}{B} L + \frac{P_o}{B} x \right) \sin \frac{n\pi x}{2L} dx$$

The Fourier coefficient then reduces to

$$b_n = \frac{-1}{L} \frac{P_o}{B} \int_0^L x \sin \frac{n\pi x}{2L} dx + \frac{1}{L} \frac{P_o}{B} \int_L^{2L} (x - 2L) \sin \frac{n\pi x}{2L} dx$$

Rearranging the coefficient equation gives

$$b_n = \frac{1}{L} \frac{P_o}{B} \int_L^{2L} x \sin \frac{n\pi x}{2L} dx - \frac{1}{L} \frac{P_o}{B} \int_0^L x \sin \frac{n\pi x}{2L} dx - 2 \frac{P_o}{B} \int_L^{2L} \sin \frac{n\pi x}{2L} dx$$

The first two integrals can be solved with Integration by Parts:

$$\int x \sin \frac{n\pi x}{2L} dx = -x \frac{2L}{n\pi} \cos \frac{n\pi x}{2L} - \int -\frac{2L}{n\pi} \cos \frac{n\pi x}{2L} dx = -x \frac{2L}{n\pi} \cos \frac{n\pi x}{2L} + \frac{2L}{n\pi} \int \cos \frac{n\pi x}{2L} dx$$

This reduces to

$$\int x \sin \frac{n\pi x}{2L} dx = \left(\frac{2L}{n\pi} \right)^2 \sin \frac{n\pi x}{2L} - x \frac{2L}{n\pi} \cos \frac{n\pi x}{2L}$$

The Fourier coefficient is then

$$\begin{aligned}
 b_n &= \frac{1}{L} \frac{P_o}{B} \left[\left(\frac{2L}{n\pi} \right)^2 \sin \frac{n\pi x}{2L} - x \frac{2L}{n\pi} \cos \frac{n\pi x}{2L} \right]_L^{2L} \\
 &\quad - \frac{1}{L} \frac{P_o}{B} \left[\left(\frac{2L}{n\pi} \right)^2 \sin \frac{n\pi x}{2L} - x \frac{2L}{n\pi} \cos \frac{n\pi x}{2L} \right]_0^L \\
 &\quad + 2 \frac{P_o}{B} \left[\cos \frac{n\pi x}{2L} \right]_L^{2L} \\
 &= \frac{1}{L} \frac{P_o}{B} \left[0 - \frac{4L^2}{n\pi} \cos n\pi - \left(\frac{2L}{n\pi} \right)^2 \sin \frac{n\pi}{2} + \frac{2L^2}{n\pi} \cos \frac{n\pi}{2} \right] \\
 &\quad - \frac{1}{L} \frac{P_o}{B} \left[\left(\frac{2L}{n\pi} \right)^2 \sin \frac{n\pi}{2} - \frac{2L^2}{n\pi} \cos \frac{n\pi}{2} \right] \\
 &\quad + 2 \frac{P_o}{B} \left[\cos n\pi - \cos \frac{n\pi}{2} \right]
 \end{aligned}$$

This reduces to

$$b_n = -\frac{8L}{n\pi} \frac{P_o}{B} \sin \frac{n\pi}{2}$$

The initial condition can then be expressed as

$$u_{(x,0)} = \sum_{n=1}^{\infty} -\frac{8L}{n\pi} \frac{P_o}{B} \sin \frac{n\pi}{2} \sin \frac{n\pi x}{2L} = -\frac{8L}{\pi} \frac{P_o}{B} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi}{2} \sin \frac{n\pi x}{2L}$$

The even terms of the series, $n = 2, 4, 6, \dots$, are equal to zero. Expanding the infinite series then gives

$$\sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi}{2} \sin \frac{n\pi x}{2L} = \sin \frac{\pi x}{2L} - \frac{1}{3} \sin \frac{3\pi x}{2L} + \frac{1}{5} \sin \frac{5\pi x}{2L} - \frac{1}{7} \sin \frac{7\pi x}{2L} + \dots$$

Re-expressing the series so that even terms are non-zero, gives

$$\sum_{n=1}^{\infty} \frac{-\cos n\pi}{2n-1} \sin \frac{(2n-1)\pi x}{2L} = \sin \frac{\pi x}{2L} - \frac{1}{3} \sin \frac{3\pi x}{2L} + \frac{1}{5} \sin \frac{5\pi x}{2L} - \frac{1}{7} \sin \frac{7\pi x}{2L} + \dots$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{-\cos n\pi}{2n-1} \sin \frac{(2n-1)\pi x}{2L} = \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi}{2} \sin \frac{n\pi x}{2L}$$

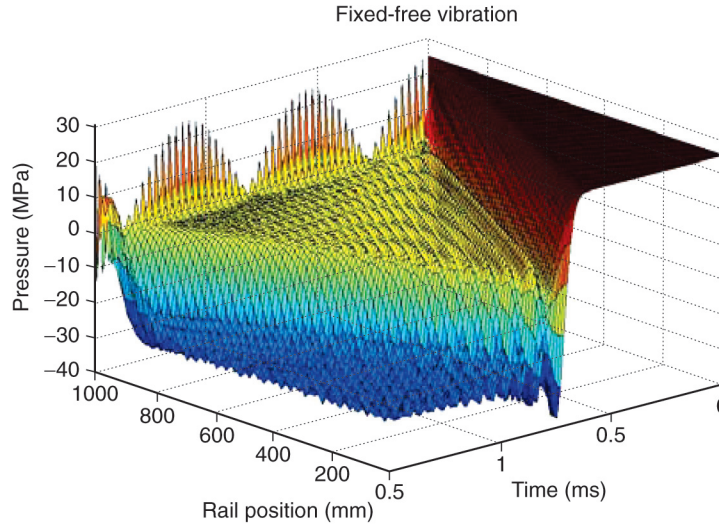


Figure 18 Response of a Fixed-Free system.

The initial condition can then be expressed as

$$u_{(x,0)} = \frac{8L}{\pi} \frac{P_o}{B} \sum_{n=1}^{\infty} \frac{\cos n\pi}{2n-1} \sin \frac{(2n-1)\pi x}{2L}$$

And the general solution can be expressed as

$$u_{(x,t)} = \frac{8L}{\pi} \frac{P_o}{B} \sum_{n=1}^{\infty} \frac{\cos n\pi}{2n-1} \sin \frac{(2n-1)\pi x}{2L} \cos \omega t$$

Finally, the pressure can then be expressed as

$$P = -B \frac{\partial u}{\partial x} = -4P_o \sum_{n=1}^{\infty} \cos n\pi \cos \frac{(2n-1)\pi x}{2L} \cos \omega t$$

A surface plot of the output from the Matlab simulation is shown in Figure 18. The large negative velocity is a result of the free end having no applied resistance. This pressure response would only be possible with vibration of a steel bar, because the pressure in a liquid would drop below vapor pressure and then stop decreasing.

A 2-d plot of the results from the Matlab simulation, overlaid with the results from the analytical expression for the fixed-free system is shown in Figure 19. The analytical solution was carried out to the first 20 terms in the infinite series. The solution was calculated in 0.2 mS time steps over the length of the rail segment. The plot shows good agreement between the analytic and numeric results. The analytic solution is calculated in the file Fixed-Free.xls.

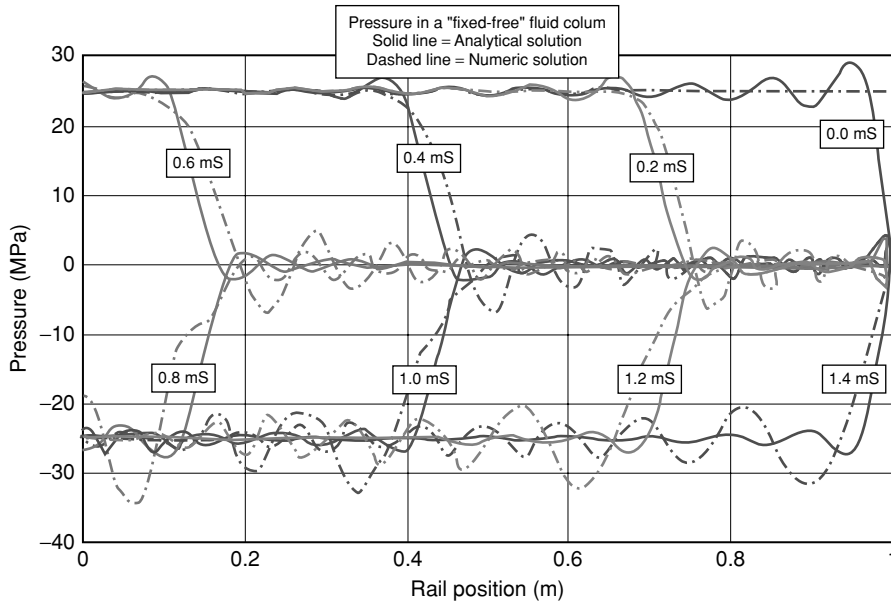


Figure 19 Comparison of analytical and numeric results.

The solution to the fixed-fixed system is somewhat simpler. As with the fixed-free system, a general solution of the wave equation can be expressed as:

$$u_{(x,t)} = \left(A \sin \frac{\omega}{c} x + B \cos \frac{\omega}{c} x \right) (C \sin \omega t + D \cos \omega t)$$

With both ends of the rail fixed, the boundary conditions are $u_{(0,t)} = 0$ and $u_{(L,t)} = 0$.

Substituting the first condition into the general solution gives

$$u_{(0,t)} = B(C \sin \omega t + D \cos \omega t) = 0 \therefore B = 0$$

Substituting the second condition into the general solution gives

$$u_{(L,t)} = \left(A \sin \frac{\omega}{c} L \right) (C \sin \omega t + D \cos \omega t) = 0$$

$$\therefore \sin \frac{\omega}{c} L = 0 \therefore \frac{\omega}{c} L = \pi, 2\pi, 3\pi, \dots \therefore \omega = \frac{n\pi c}{L}$$

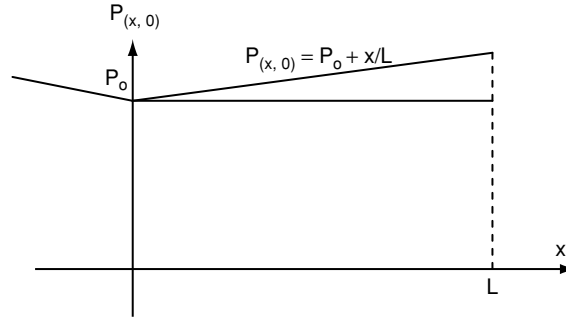


Figure 20 Fixed-Fixed initial condition.

The first initial condition is $v_{(x,0)} = 0$

The velocity at time zero is

$$\dot{u}_{(x,0)} = AC\omega \sin \frac{\omega}{c} x = 0 \therefore C = 0$$

The general solution can then be rewritten in the form

$$u_{(x,t)} = u_o \sin \frac{\omega}{c} x \cos \omega t$$

The pressure can then be expressed as

$$P = -B \frac{\partial u}{\partial x} = -B \frac{\omega}{c} u_o \cos \frac{\omega}{c} x \cos \omega t$$

where B is the Bulk Modulus of the fluid. The total solution is then the sum of all possible solutions,

$$p = -\frac{B}{c} \sum_{n=1}^{\infty} \omega u_o \cos \frac{\omega}{c} x \cos \omega t$$

The constant u_o can be evaluated from the remaining initial condition.

Figure 20 is a graph of a simple initial condition that works well for demonstration of the solution.

For an even Fourier series,

$$f_{(x)} = a_o + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} \text{ where } a_o = \frac{1}{L} \int_0^L f_{(x)} dx \text{ and } a_n = \frac{2}{L} \int_0^L f_{(x)} \cos \frac{n\pi x}{L} dx$$

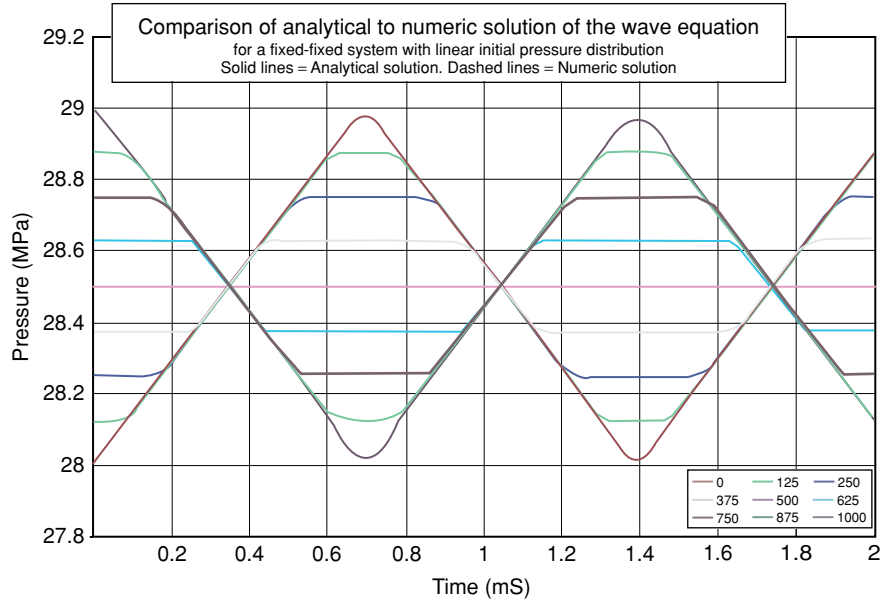


Figure 21 Fixed-fixed system output.

For the function in Figure 20, the offset constant is

$$a_o = \frac{1}{L} \int_0^L \left(P_o + \frac{x}{L} \right) dx = \frac{1}{L} \left[P_o x + \frac{x^2}{2L} \right]_0^L = \frac{1}{L} \left(P_o L + \frac{L}{2} \right) = P_o + \frac{1}{2}$$

And the series constant is

$$\begin{aligned} a_n &= \frac{2}{L} \int_0^L \left(P_o + \frac{x}{L} \right) \cos \frac{n\pi x}{L} dx \\ &= \left[\frac{2P_o}{L} \frac{L}{n\pi} \sin \frac{n\pi x}{L} \right]_0^L + \frac{2}{L^2} \int_0^L x \cos \frac{n\pi x}{L} dx \\ &= \left[\frac{2P_o}{n\pi} \sin \frac{n\pi x}{L} \right]_0^L + \frac{2}{L^2} \left[\frac{Lx}{n\pi} \sin \frac{n\pi x}{L} - \int_0^L \frac{L}{n\pi} \sin \frac{n\pi x}{L} dx \right] \\ &= \frac{2}{L^2} \left[\frac{Lx}{n\pi} \sin \frac{n\pi x}{L} + \left(\frac{L}{n\pi} \right)^2 \cos \frac{n\pi x}{L} \right]_0^L \end{aligned}$$

Which reduces to

$$a_n = \frac{2}{(n\pi)^2} (\cos n\pi - 1)$$

The initial pressure is then

$$P_{(x,0)} = P_o + \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{(n\pi)^2} (\cos n\pi - 1) \cos \frac{n\pi x}{L}$$

And the total solution is the superposition of the offset (a_o) and the infinite series

$$P_{(x,t)} = P_o + \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{(n\pi)^2} (\cos n\pi - 1) \cos \frac{n\pi x}{L} \cos \omega t$$

Figure 21 is a comparison of the analytic results to the numeric results. Again in this case the analytic and numeric results show good agreement. The analytical solution was carried out for the first four terms of the infinite series. The results for this system are calculated in the file Fixed Fixed.xls.

4. CONCLUSIONS

1. The pressure waves move through the rail at the wave speed. The actual wave speed is 1440 m/S. This can be verified from Figure 19 where the wave traverses the rail (1 m) in approx 0.7 mS. The velocity for the wave would then be

$$v = \frac{\Delta x}{t} = \frac{1m}{0.7mS} = 1429 \text{ m / S}$$

2. The disturbances in the pressure distribution, due to injector and pump flow, propagate back and forth inside of the rail, reflecting off of the ends.
3. The pressure oscillations are larger than what is seen in a real physical system. This is probably due to the omission of damping effects in the rail.
4. An initial pressure distribution in the rail superimposes onto the pressure distribution that is generated by the next injection.
5. The pressure variation is not sufficient to cause emissions problems at Engine Idle or HSL, but are sufficient to cause problems at Peak Torque.

RECOMENDATIONS

1. Use output from the model to generate pressure disturbances due to injection and pump flow, and apply these “sub-solutions” as initial conditions to the d’Alembert solution. It may then be possible to “pseudo-analytically” solve the system, continuously from one injection to the next.
2. Repeat the analysis with damping built into the model.
3. Add non-linearity for the spool, needle and RFC.
4. Couple the injector and rail models to give a more accurate ROI boundary condition.

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