Exact code scaling

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ABSTRACT

A new possibility of code scaling is introduced. We show that there are not extreme volumes over which some equations of mathematical physics have the same eigenvalues and there exists a simple transplantation rule to get the eigenfunction of the first volume once that of the second volume is known. We present two techniques. In the first, the domain of the numerical method is a discretized volume. Congruent elements are glued together to get the domain over which the solution is sought. We associate a group and a graph to that volume. When the group is a symmetry of the boundary value problem, one can specify the structure of the solution, and predict the existance of other equispectral volumes. The second technique uses a complex mapping to transplant the solution from volume V_1 to volume V_2 and a correction function. Equation for the correction function is given. A simple example demonstrates the feasibility of the suggested method. We show that a measurement associated with the fundamental eigenfunction of a linear operator on a volume is sufficient to predict results of a measurement on another volume by a computer program.

1. INTRODUCTION

In the design and safety analysis of large industrial devices, calculational models are tested against experiments carried out on a small scale mock-up. This is the case with nuclear power plants [1], aeroplanes [2], and ships [3]. We would need a transplantation of the measured values to the geometry of the real scale device. Is there any hope of doing the transplantation exactly or we have to put up with approximate methods [4]? Is there a chance to achieve exact code scaling?

The finite element method is widely used in modeling industrial devices. In that method, the analyst subdivides the volume under consideration into a large number of subvolumes called elements. That task is so complex that computer programs are used to prepare the discretization. Efficiency of the finite element method depends on the selected discretization. But how could the analyst comprehend a discretization scheme that is so complicated that only a computer program is capable of generating it? We suggest an algebraic description of the discretization and our description makes it possible for the analyst to compare discretization schemes and find out if two discretizations are essentially the same.

In a finite element calculation, the analyst may use different discretizations for the same volume. Presumably the analyst would prefer that discretization which enhances the

phenomenon to be investigated. Then all discretizations refer to the same volume. In code scaling the situation is different. We are looking for two volumes that are not equivalent, hence there is no simple rotation, reflection or translation that would bring one into the other. When we find such volumes one of them serves as the volume where measurements are carried out the other as the volume of the real device. We show that using the algebraic description of the discretization such volumes can be generated. Let the mentioned volumes be V_1 and V_2. If we can find a transformation that brings the solution on V_1 into the solution on V_2, we have created an example. Let V_1 be the volume where measurement is carried out, then we can transplant the solution to V_2. In that case the code scaling is exact: we are able to transform the measured values from V_1 to V_2 without any approximation. If that can be done, there is a hope for a more precise uncertainty estimation for models of complex systems, such as a nuclear power plant. We show that there exist equispectral, discretized volumes V_1 and V_2 which are not trivially transformable into each other.

2. CODE SCALING TODAY

In studying large industrial devices, mock-ups, scaled down devices have been used for a long time. The measurements are transferred to the real size facility by a scaling law. Such scaling recipes are more than a century old. In hydrodynamics, the basic equations are non-linear [5], therefore the scaling is restricted to specific "characteristic" distances, velocities etc. Various scaling methods are used even today, for example in the nuclear industry [6] and in code validation [7].

The code scaling originates from simple transformations of the units by which basic physical quantities (length, time, and mass) are measured. In thermal hydraulics, where the measurements are carried out on a *model* and are used to assess the accuracy of the calculation of a *real device*, the method can be summarized as follows [8]:

1. The basic conservation equations (material, momentum and energy conservations) are put down. We also need the state equation which involves the pressure, material density, and enthalpy. The equations are put down for the *model* and the *real device*.
2. The *model* and the *real device* are characterized by the following parameters:
   a. ratio of characteristic lengths
   b. ratios of characteristic times, velocities and accelerations
   c. ratio of heat generation per unit volume.
3. By comparing the conservation equations for the *model* and the *real device*, a set of relationships is obtained among the parameters mentioned in item 2.

As a conclusion, at the end of their work, Nahavandi, Castellana and Moradkhanian [9] arrive at three scaling laws called time reducing, and two versions of time preserving scales. The scaling law is a set of expressions [10] to be preserved by the model (for details which are of interest only for experts in heat and mass transfer see Ref. [10]), among others:

- Richardson number
- Friction number
- Modified Stanton number
- Biot number
- Reynolds number.

When the *model*, which is a scaled down experimental facility, and the *real device*, which might be as complex as a nuclear power plant, have identical Richardson number, friction number etc. one may hope for a transplantation recipe. But this is not the case. The recipe

provided by code scaling is global in the sense that transplants properly only overall features of the thermal hydraulical phenomena, not the detailed solution as misbelieved by some. Although the derived simple relations have proved rather useful, for example in designing static states of *model*s [10] but the transplantation of detailed solution has not been achieved. The difficulty is in the non-homogeneous nature of the phenomena under investigation. In a part of the device water, in another steam, in a third one a mixture of steam and water flows. Those phenomena can not be transplanted by a single rule. Below we formulate the code scaling or transplantation problem.

The principal problem of code scaling is formulated as follows. We need a model for a large and expensive device. To study the relevant physical processes, to verify the performance of the model we carry out measurements on a scaled down model. In a basic report of the nuclear industry [10], the motivation is formulated in the following way: *"It is obvious that full scale experiments will be too dangerous and costly, thus it is necessary to use scale models and simulation experiments."* Similarity principles are utilized in the design of the experimental scaled down facility and also in the transplantation of the measurements to the real device. As Ishii and Kataoka write [10] : "*Similarity laws and scaling criteria are quite important for designing, performing, and analyzing simulation experiments using a scale model."*

The mathematical aspect of the problem is phrased as follows. The physical processes are governed by an equation:

$$\mathbf{A}(\underline{x})\Psi(\underline{x},\ p) = \lambda\Psi(\underline{x},\ p),\ \underline{x} \in V_1 \tag{1}$$

where operator $\mathbf{A}(\underline{x})$ implicitely depends on a set of parameters p, acts on the local space variables $\underline{x}$, and the physical process is characterized by the eigenfunction $\Psi(\underline{x},\ p)$. We assume that Eq. (1) unequivocally determines the state function $\Psi(\underline{x},\ p)$. The device is finite, at its boundary a suitable, homogeneous boundary condition is prescribed.

We formulate the problem of code scaling as follows. We are looking for a domain V_2 such that

$$\mathbf{B}(\underline{x}')\Phi(\underline{x}', p') = \lambda\Phi(\underline{x}', p')\ \underline{x}' \in V_2. \tag{2}$$

We assume the existence of a one-to-one map $\mathbf{T}: V_1 \to V_2$. When we wish to emphasize that $\mathbf{T}$ brings a point $\underline{x} \in V_1$ into a point $\underline{x}' \in V_2$ we write $\mathbf{T}$: $\underline{x} \to \underline{x}'$, or $\underline{x} = f(\underline{x}')$ where f is a given function. Operator $\mathbf{B}(\underline{x}')$ acts on the local space variable $\underline{x}'$ and is obtained from $\mathbf{A}(\underline{x})$ by the substitution $\underline{x} = f(\underline{x}')$. The parameters are also affected by the map $\mathbf{T}$ because diameters, lengths (geometrical data in general) are also among the parameters. Let us write the transformed parameters as p'. We need a transplantation rule $\Psi(\underline{x},\ p) \to \Phi(\underline{x}',\ p')$ to transplant the measured values from V_1 to V_2 .

3. NEW APPROACH TO CODE SCALING

The invariance properties of the involved equations have been studied [11,12] for almost two decades but only the advent of computer codes has made it possible to extend the set of problems being studied. Such analyses have revealed that the symmetry groups of the conservation equations, at least in their Boussinesq form [13], have only the Euclidean group *E(2)* as their symmetry in plane geometry. No group associated with the above mentioned scaling method has been found. The reason is in that code scaling formulates not precise invariance observed at each point but in global (therefore approximate) large scale relationships.

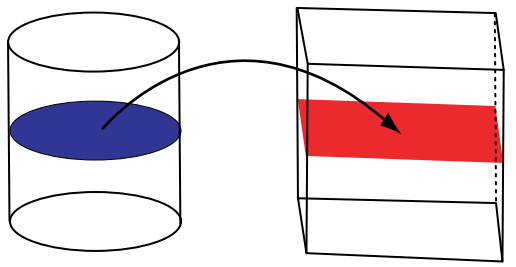

Figure 1 Mapping a three-dimensional region into another.

On the other hand, studies of the simplest equations, such as the eigenvalue problem of the Laplace operator in a homogeneous volume, have made it clear that the transformations leaving the equation invariant produce only trivial equivalent volumes because those transformations include only translations, rotations and reflections.

Recent achievements of computational group theory and modern algebra discovered volumes which are equispectral but there is no symmetry of the equation that would transform one volume into the other one. We point out that for engineering practice equispectrality is often too strong, one would put up with volumes with identical fundamental eigenvalues accompanied with a transplantation recipe of the fundamental eigenfunctions.

3.1. MAPS AND EQUATIONS

The present section deals with the relationship of transformation rules called maps and equations describing a given physical phenomenon. For us, the equation is an operation (such as differentiation, integration) to be applied to the solution. We use the term operator for those operations. The transformation is regarded as a functional relationship between the new coordinates and the old ones. We recall that the $\mathbf{T}$: $\underline{x} \rightarrow \underline{x}'$ notation describes a transformation bringing a point $\underline{x}$ in V_1 into a point $\underline{x}'$ in V_2.

In the discourse, we are dealing with two-dimensional (2D) problems. The reason is the following. Let us consider the map $V_1 \rightarrow V_2$. By cutting V_1 and V_2 with a plane, perpendicular to the z axis, we get two, two-dimensional domains D_1 and D_2 marked by blue (dark) and red (grey) colors. If we find a map $f: D_1 \rightarrow D_2$ shown in Fig. 1, we can construct the map $f(z): V_1 \rightarrow V_2$.

3.2. PASTING

Below we point out that there exist equispectral volumes which are not transformed into each other by any symmetry of the investigated equation. In most physical problems, the phenomenon to be studied is discribed by an eigenvalue problem. Hence, the starting point of our analysis is a general eigenvalue problem. First, we derive a formal solution to the boundary value problem under consideration.

Let us investigate the following eigenvalue problem in V:

$$\mathbf{A}\Psi(\underline{x}) = \lambda\Psi(\underline{x}),\ \underline{x} \in V \tag{3}$$

where operator $\mathbf{A}$ is such that

$$\left[\mathbf{A},\mathbf{O}\right] = \mathbf{AO} - \mathbf{OA} = 0. \tag{4}$$

Here **O** is a symmetry (or automorphism) of V, viz. a reflection, translation or rotation operator acting on functions defined over V. Eq. (4) is the usual definition of symmetries of operator **A**. When seeking a V_1 that can be transformed by a map **T** into another volume V_2, we have to construct that volume. In the construction, we apply only symmetries of operator **A** to a tile (or building block) t. The symmetry group of **A** is known to be isomorphic to a group of transformations commuting with **A**. It is well known that the solution of eigenvalue problem (3) with a homogeneous boundary condition along the boundary ∂V of V is easily determined from a solution of the same problem over another volumes which can be transformed into V by symmetries. For example, if we have the solution in a volume which is just V but displaced, we know that the solution should be the same as in V, provided **A** is invariant under displacement. The question is, if we can find a volume V_2 not equivalent[1] to V_1 such that all the eigenvalues of problem (3) will remain the same as for V_1. In our investigations, V_1 is constructed by gluing together copies of a tile t. Such a structure can be obtained by reflections, an assumed symmetry of Eq. (3). V_1 composed in that way is characterized by the connection of copies of t. This is described by a so called adjacency matrix **U**, and $U_{ij} = 1$ if copy i and j share an edge, otherwise $U_{ij} = 0$.

Below we derive a formal solution [14] to problem (3) with Dirichlet boundary condition (i.e. $\Psi = 0$ on the boundary). The solution is given in terms of the Green's function G_t of tile t, that we obtain as the solution of the following boundary value problem:

$$(\mathbf{A} - \lambda)G_t(\underline{x},\xi) = 0,\ \underline{x} \in t \tag{5}$$

$$G_t(\underline{x},\xi) = \delta(\underline{x} - \xi),\ \xi \in \partial t. \tag{6}$$

Here ft is the boundary of tile t and δ is Dirac's delta function. Solution of

$$\mathbf{A}\Psi(\underline{x}) = \lambda\Psi(\underline{x}),\ \underline{x} \in t \tag{7}$$

with boundary condition

$$\Psi(\underline{x}) = \begin{cases} f_a(\xi), \xi \in \partial t_a \\ f_b(\xi), \xi \in \partial t_b. \\ f_c(\xi), \xi \in \partial t_c \end{cases} \tag{8}$$

(here the three sides of tile t are ∂t_a, ∂t_b and ∂t_c) is

$$\Psi(\underline{x}) = \int_{\partial t_a} G_t(\underline{x},\xi) f_a(\xi) d\xi + \int_{\partial t_b} G_t(\underline{x},\xi) f_b(\xi) d\xi + \int_{\partial t_c} G_t(\underline{x},\xi) f_c(\xi) d\xi,\ \underline{X} \in \mathrm{t}. \tag{9}$$

[1] V_1 and V_2 are equivalent if a symmerty of operator A transforms V_1 in to V_2.

Since a volume V (which may be V_1 or V_2) consists of copies of t, the solution of Eq. (1) in V is the sum of integrals like Eq. (9). Let us assume that there are K internal sides[2] and the solution is $f_i(\xi)$along side i. We compose $\Phi(x)$ from N functions $\mathbf{F}(\underline{x})=(\Phi_1(\underline{x}),\ldots,\Phi_N(\underline{x}))$, one from each copy of t. Then,

$$\mathbf{F}(\underline{x}) = \mathbf{Q}\mathbf{v}(\underline{x}), \tag{10}$$

where

$$v_k(\underline{x}) = \int_{\partial t_k} G_t(\underline{x},\xi) f_k(\xi) d\xi, k = 1,\ldots,K \tag{11}$$

$\mathbf{Q}$ is an $N\times K$ matrix[3], and assigns sides to copies of t: $Q_{ik}=1$ when copy i has internal boundary $0\le k\le K$ as an edge. As we see, expression (10) has two components, $\mathbf{v}(\underline{x})$ depends only on tile t, operator $\mathbf{A}$, and eigenvalue λ, hence we call it the physical part of the solution. On the other hand, $\mathbf{Q}$ depends only on the structure of V, hence we call it the structural part of the solution. Structure (10) can be used to derive an integral equation set for the solution along the internal boundaries and serves as basis for approximate solution methods. Now we are interested in the structural part. We introduce a so called auxiliary matrix $\mathbf{X}$ which is constructed as follows. In the i^{th} row, its elements are 1 in the positions corresponding to neighboring copies of t with copy i, and, the diagonal element gives the number of internal boundaries of copy i.

The auxiliary matrix $\mathbf{X}$ of volume V and the structural matrix $\mathbf{Q}$ in Eq. (8) are related as $\mathbf{X}=\mathbf{Q}\mathbf{Q}^+$, where $\mathbf{Q}^+$ is the transpose of $\mathbf{Q}$.

The advantage of the introduced new matrices comes clear from the following claim. Let $\mathbf{X}_i$ be the auxiliary matrix of discretized volume V_i, i = 1,2 . If the eigenvalues of matrices $\mathbf{X}_i$ are the same then V_1 and V_2 are isospecral.

It can be shown that there are discretized volumes which are isospectral. In Fig. 2, we present two, equispectral, and discretized volumes composed of seven triangles.

Once we have isospectral volumes, the solutions of the eigenvalue problems are transformed into each other by a linear transformation. The transformation recipe is: $\mathbf{F}_2(\underline{x})=\mathbf{M}\mathbf{F}_1(\underline{x})$, when $\mathbf{X}_1=\mathbf{M}\mathbf{X}_1\mathbf{M}^+$. Remember, here $\mathbf{F}_i(\underline{x})$ denotes a vector, its components correspond to the copies of tile t in V_i .

3.3. COMPLEX MAPS

Let us introduce local coordinates in V_1 and V_2 as follows. A point in V_2 and V_1 is $\underline{x}'=(x',y')$ and $\underline{x}=(x,y)$, respectively. The map $\mathbf{T}$: $V_1\to V_2$ be given by the coordinate transformations

$$x' = \varphi(x,y) \tag{12}$$

$$y' = \theta(x,y). \tag{13}$$

There is a theorem by Riemann according to which every simply connected V_1 and V_2 are joined by an invertible map. That invertible map has been constructed through complex

[2] A side is called internal if it is a common boundary of two copies fo t.

[3] Since every reflection creates an internal side, $K = N$-1.

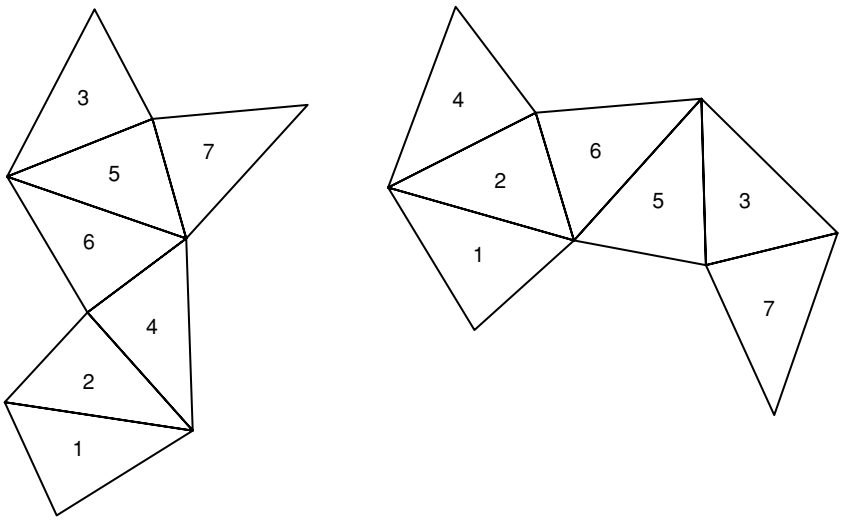

Figure 2 Equispectral discretized volumes composed of 7 triangles.

functions [11]. Let $z = x + iy$ and $z' = x' + iy'$ where $i = \sqrt{-1}$. Then, $z' = f(z)$ is the complex function realizing the map $V_1 \rightarrow V_2$. A short introduction to relationship between the applied groups, graphs and maps is given in the Appendix.

The operators occurring in Eq. (3), act on variables (x,y). Transformation **T** makes the operations to act on variables (x',y'), the transformed operator is written as **TA**. We say that transformation **T** is symmetry of **A** when it leaves **A** unchanged[4]. Then **A** acting on coordinates $\underline{x}$=(x,y) is the same as acting on coordinates (x',y'), in that case we have **AT**=**TA**. If $\mathbf{T}_1$ and $\mathbf{T}_2$ leave **A** invariant then $\mathbf{T}_1\mathbf{T}_2$ also leaves **A** invariant. In other words, the symmetries of **A** form a group.

The symmetries of various equations have been studied [11] and for example the symmetries of the Laplace operator include translations, rotations and reflections. Those transformations result in a volume equivalent to the original one hence for us they are irrelevant. We exploit the complex mapping so that we seek the solution of the problem

$$\mathbf{A}\Phi(\underline{x}') = \lambda\Phi(\underline{x}'), \underline{x}' \in V_2 \tag{14}$$

in the following form:

$$\Phi(\underline{x}') = \mathbf{T}(\Psi)(\underline{x}) + p(\underline{x}') \tag{15}$$

and we are to derive an equation for the correction function $p(\underline{x}')$. Substituting expression (15) into the eigenvalue problem in V_2 and exploiting the linearity of **A**, we get

[4] Then the forms of operator A in variables (x_1, y_1) and (x_2, y_2) are the same.

$$\mathbf{AT}\,\Psi(\underline{x}) + \mathbf{A}p\,(\underline{x}') = \lambda\mathbf{T}\Psi(\underline{x}) + \lambda\,p(\underline{x}'), \underline{x}' \in V_2, \underline{x} \in V_1$$
$$(\mathbf{A} - \lambda)p(\underline{x}') = (\lambda - \mathbf{A})\mathbf{T}\Psi(\underline{x}), \underline{x}' \in V_2, \underline{x} \in V_1$$

we obtain the following non-homogeneous equation:

$$(\mathbf{A} - \lambda)\,p(\underline{x}') = Q(\underline{x}'), \tag{16}$$

where the source term is $Q(\underline{x}') = -(\mathbf{A}-\lambda)\mathbf{T}\Psi(\underline{x})$. The mathematical form of the eigenvalue problem is the same in V_1 or V_2. But the physical contents of the two problems are radically different. It may happen that a measurement can not be performed on V_2, or, the measurement is very expensive. If we have a feasible transformation process then it suffices to measure $\Psi(\mathbf{x})$ in V_1 and using a computer program that solves Eq. (16) we get the measured values in V_2. The source term is known in Eq. (16), since in the source every component refers to volume V_1, and the numerical (or, in exceptional cases analytical) solution of Eq. (16) causes no problem.

The procedure proposed above is applicable only to linear operators; in the general case Eq. (16) is more complicated. In the next section, we present a transplantation of the diffusion equation from square shaped region to a circular region.

4. APPLICATION

Let us consider the boundary value problem (BVP) in domain V_1 as

$$\mathbf{A}\Psi(\underline{x}) = \lambda\Psi(\underline{x}),\ \underline{x} \in V_1 \tag{17}$$

and boundary condition

$$\Psi(\underline{x}) = 0,\ \underline{x} \in \partial V_1. \tag{18}$$

In domain V_2 , we get a transformed operator, which may differ from **A**:

$$\mathbf{B}\Phi(\underline{x}') = \lambda\Phi(\underline{x}'),\ \underline{x}' \in V_2 \tag{19}$$

and boundary condition

$$\Phi(\underline{x}') = 0,\ \underline{x}' \in \partial V_2. \tag{20}$$

Here **A** and **B** are linear operators, and **B** is obtained[5] by the $\underline{x} \to \underline{x}'$ substitution in operator **A**. Let the one-to-one map **T**: $V_1 \to V_2$ be given. Then

$$\Phi(\underline{x}') = [\mathbf{T}(\psi)](\underline{x}) + p(\underline{x}') \tag{21}$$

[5] Note that in general $\mathbf{A} \neq \mathbf{B}$.

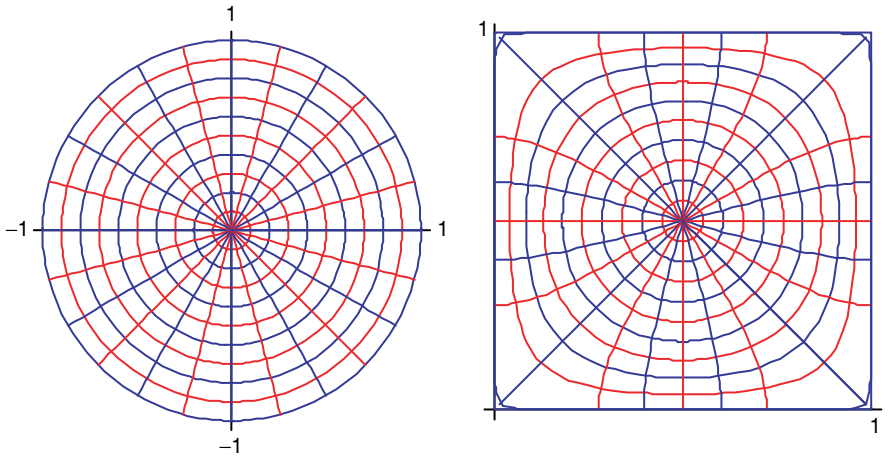

Figure 3 The unit disk and the unit square. Colors indicate transformed niveau lines of the solution.

and the correction function $p(\underline{x}')$ meets the following equation:

$$(\mathbf{B}-\lambda)\,p(\underline{x}') = Q(\underline{x}'),\ \underline{x}' \in V_2, \tag{22}$$

and the source $Q(\underline{x}')$ is given by

$$Q(\underline{x}') = -(\mathbf{B}-\lambda)[\mathbf{T}(\Psi)](\underline{x}'). \tag{23}$$

We assume that Ψ is measured easily but it would be expensive to measure Φ. Using (22)-(23), we predict Φ from Ψ. First we obtain the correction function $p(\underline{x}')$, after that we use Eq. (21) to get Φ. In order to carry out the mentioned steps, we need the transformation rule $\mathbf{T}$ and a numerical solution of Eq. (22).

In the example given below, we show how to transform the solution of the one energy group, neutron diffusion equation (its mathematical form is reduced to $\Delta f - k^2 f = 0$, where k is a constant), over a square region, the solution is known to be $\Psi(\mathbf{x}')$ into the solution on a disk denoted by $\Phi(\underline{x})$. Since both known solutions are analytical, the accuracy of the method can readily be tested.

The transformation relating the unit disk and a square region with unit sides can be found with the Schwarz-Christoffel method [15]. We start from the disk (shown in Fig. 3) and apply the following transformation to get the square (apparently, variables x and y are treated as the real and imaginary part of a complex variable z).

We know the solution on the square to be composed of normalized sinus functions:

$$\Psi(\underline{x}') = \sqrt{\frac{2}{l}} \cdot \sin\left(\frac{\pi \cdot \mathrm{Re}[x'+iy']}{l}\right) \cdot \sqrt{\frac{2}{m}} \cdot \sin\left(\frac{\pi \cdot \mathrm{Im}[x'+iy']}{m}\right)$$

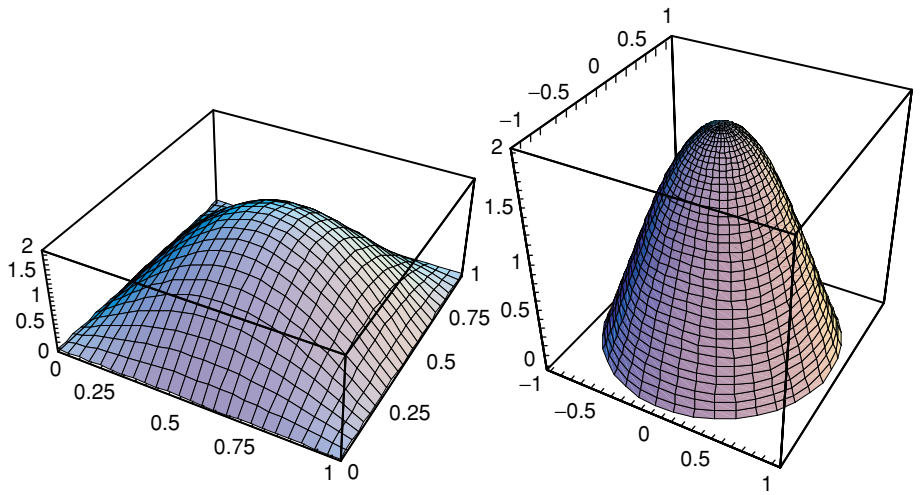

Figure 4 Solution on the unit square and its transformed version on the disk.

Thus:

$$[\mathbf{T}(\Psi)](\underline{x}) = \sqrt{\frac{2}{l}} \cdot \sin\left(\frac{\pi \cdot \mathrm{Re}[\mathbf{T}(x'+iy')]}{l}\right) \cdot \sqrt{\frac{2}{m}} \cdot \sin\left(\frac{\pi \cdot \mathrm{Im}[\mathbf{T}(x'+iy')]}{m}\right)$$

With polar coordinates ($x' = r' \cdot \cos(\varphi')$ and $y' = r' \cdot \sin(\varphi')$):

$$[\mathbf{T}(\Psi)](\underline{x}) = \sqrt{\frac{2}{l}} \cdot \sin\left(\frac{\pi \cdot \mathrm{Re}[\mathbf{T}(r'\cos\varphi' + ir'\sin\varphi')]}{l}\right) \cdot \sqrt{\frac{2}{m}} \cdot \sin\left(\frac{\pi \cdot \mathrm{Im}[\mathbf{T}(r'\cos\varphi' + ir'\sin\varphi')]}{m}\right)$$

The original solution on the square and its transformed form is depicted in Fig. 4.

Since the analytical solution of the diffusion equation is known in either geometry, the accuracy of the proposed procedure can be estimated readily. It is easy to calculate the correction term because an analytical solution can be given on the disk as well: any solution is composed of first order Bessel functions. Expanding $\mathbf{T}(\Psi)$ with the Bessel functions (Bessel zeros are the multipliers in the brackets) we get:

$$[\mathbf{T}(\Psi)](\underline{x}) = -2.0062 J_0(2.40483r) + 0.00894 J_0(5.52008r) - 0.00381 J_4(7.58834r) -$$

$$-0.00054 J_4(11.0674r) - 0.00049 J_8(12.2251r) - 0.00013 J_8(16.0378)$$

This approximation is fairly precise: the difference between the analytical solution on the disk (the first term) and the function transplanted from the disk ($\mathbf{T}(\Psi)$) is 0,14% in case of expansion with 6 terms. Those terms are comprised in the correction function. The exact

solution is in the first term, so instead of solving Eq. (22), we could test the accuracy of our method.

Combining our findings with the fact that simply connected volumes can always be mapped into each other, we find that any measurement that is interpreted as the dominant eigenfunction of linear operator **A** has to be measured only in one volume, the measured values are transformable into all other volumes by means of the techniques presented above. The transformation involves the solution of non-homogeneous problem (22).

5. CONCLUDING REMARKS

We have shown that there exist equispectral volumes such that no symmetry of the equation transforms them into each other. If one can create two volumes, V_1 and V_2, such that their auxiliary matrices have the same eigenvalues, then the solutions of the investigated equation over the two volumes are transformed into each other by a linear transformation. As C. Gordon and D. Webb pointed out [16], there exist equispectral volumes in 2D. With those volumes, one can associate a linear transplantation.

To find equispectral volumes one can use group theoretical methods. Finding three groups forming a Sunada triple, as proposed in Ref. [16], one can create discretized volumes which are equispectral. Unfortunately, little information is available on the Sunada triples in general, that makes it difficult to formulate general statements concerning discretized volumes. Another possibility is to investigate eigenvalues of the auxiliary matrixces. Even here, the trial and error approach is used in the lack of a systematic method for constructing an equispectral counterpart for a given discretized volume.

For linear operators, one may apply the correction function, which is obtained from a non-homogeneous equation, the source term being given by the transform of the known solution. In other words, one can transform the measured values to another volume using the correction function. In that case we need a $V_1 \rightarrow V_2$ map transforming the *model* volume into the *real volume*.

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APPENDIX

GROUPS, GRAPHS, MAPS

A discretized volume V is nothing else but glued together copies of a tile t. To facilitate work with discretized volumes, we render a group G and a graph Γ to a discretized volume in the following way. Let our tile be a triangle, with sides α, β, and γ. In engineering applications, it is rather difficult to assess discretized volumes because of the large number of tile copies in a discretized volume. We associate a group to the discretized volume as follows.

We associate a permutation group G to V in the following way. When in V a side of type α connects the copies $i_{\alpha 1}$ with $i_{\alpha 2}$ and $i_{\alpha 1}$ with $i_{\alpha 2}$ … then, we form the permutation a = ($i_{\alpha 1}$. ,$i_{\alpha 2}$) ($j_{\alpha 1}$, $j_{\alpha 2}$)…. We carry out that procedure for sides α, β, and γ to get generators a , b, and c, and group G is generated by a , b, and c.

A graph Γ is assigned to V in the following way. We number the copies of t in V. If copies numbered n_1 and n_2 are adjacent, and they share a side of type α , then the vertices n_1 and n_2 of graph Γ are connected by an edge of type α .

Let us consider the discretized volumes shown in Fig. 2. Let the sides of tile t be α, β, and γ increasing in length. The group associated with the discretized volume on the left is generated by a=(6,4)(5,7); b=(2,4)(3,5); c=(1,2)(6,5). The graph associated with the left figure is $1-2-4-6-5<_7^3$.

The thus far known equispectral, not equivalent volumes have been constructed using group G and graph Γ. The construction relies either on hyperbolic geometry or group theory. Interested reader find further details in Ref. [16].